Proof Complexity of Practical Integer Programming

Noah Fleming

Based on

- On the Power and Limitations of Branch and Cut

  Fleming, Göös, Impagliazzo, Pitassi, Robere, Wigderson

Outline

- Using Proof Complexity for Algorithm Analysis
- Proof Complexity of Integer Programming
  - Cutting Planes
- Branch and Cut and Stabbing Planes
- Stabbing Planes vs. Cutting Planes

Algorithm Analysis from Proofs

Idea: Formalize the techniques used in a class of algorithms A as a proof system P

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- Hides practical details of algorithms
- Lower bounds on P-proofs → lower bounds on runtime of A

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- e.g. Algorithms for SAT
  - CDCL and Resolution
- e.g. Algorithms for Integer Programming
  - Chvátal-Gomory Cutting Planes and Cutting Planes

Integer Programming

Integer-programming: Given $Ax \geq b$ find $x \in \mathbb{Z}^n, Ax \geq b$

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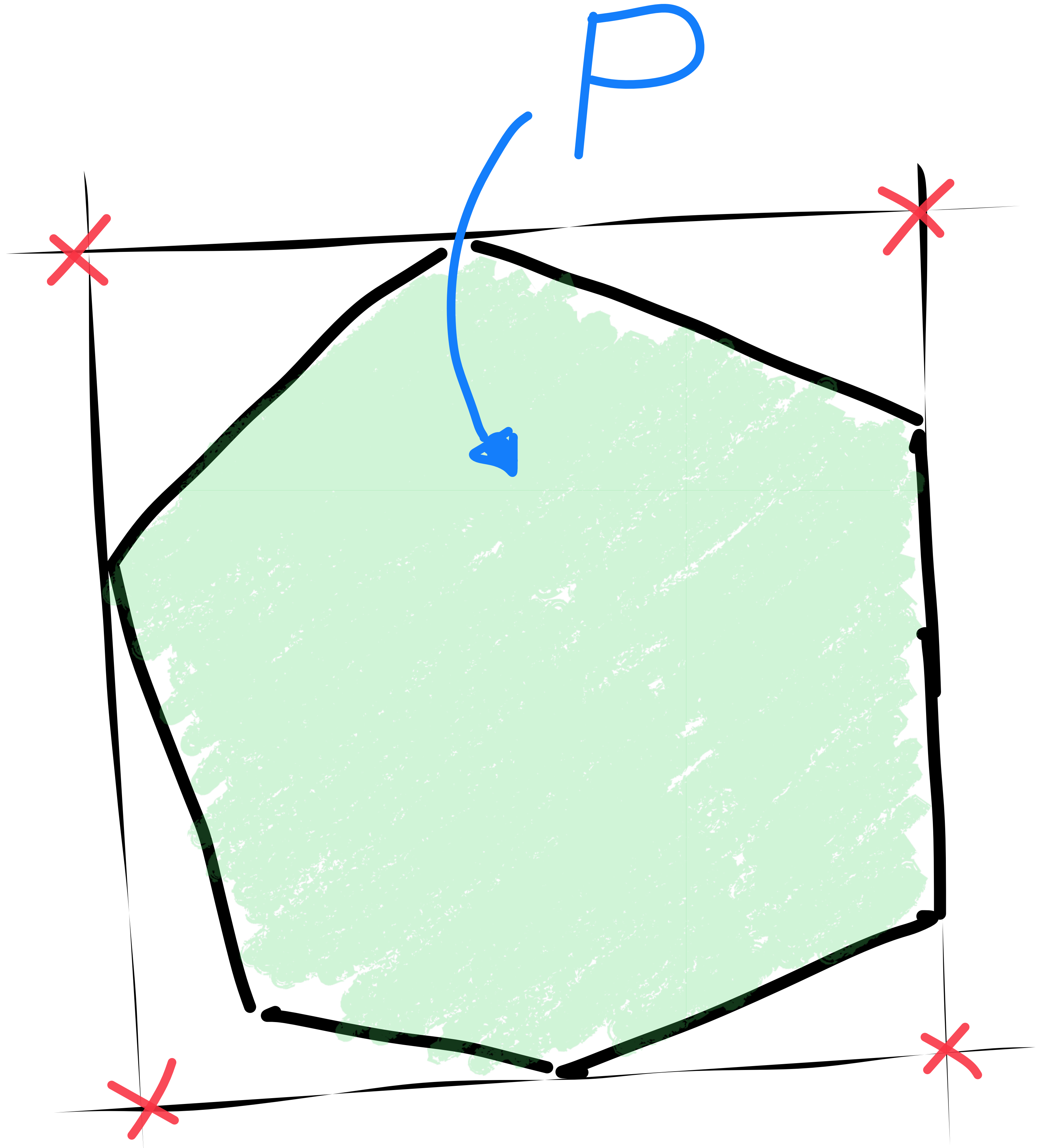

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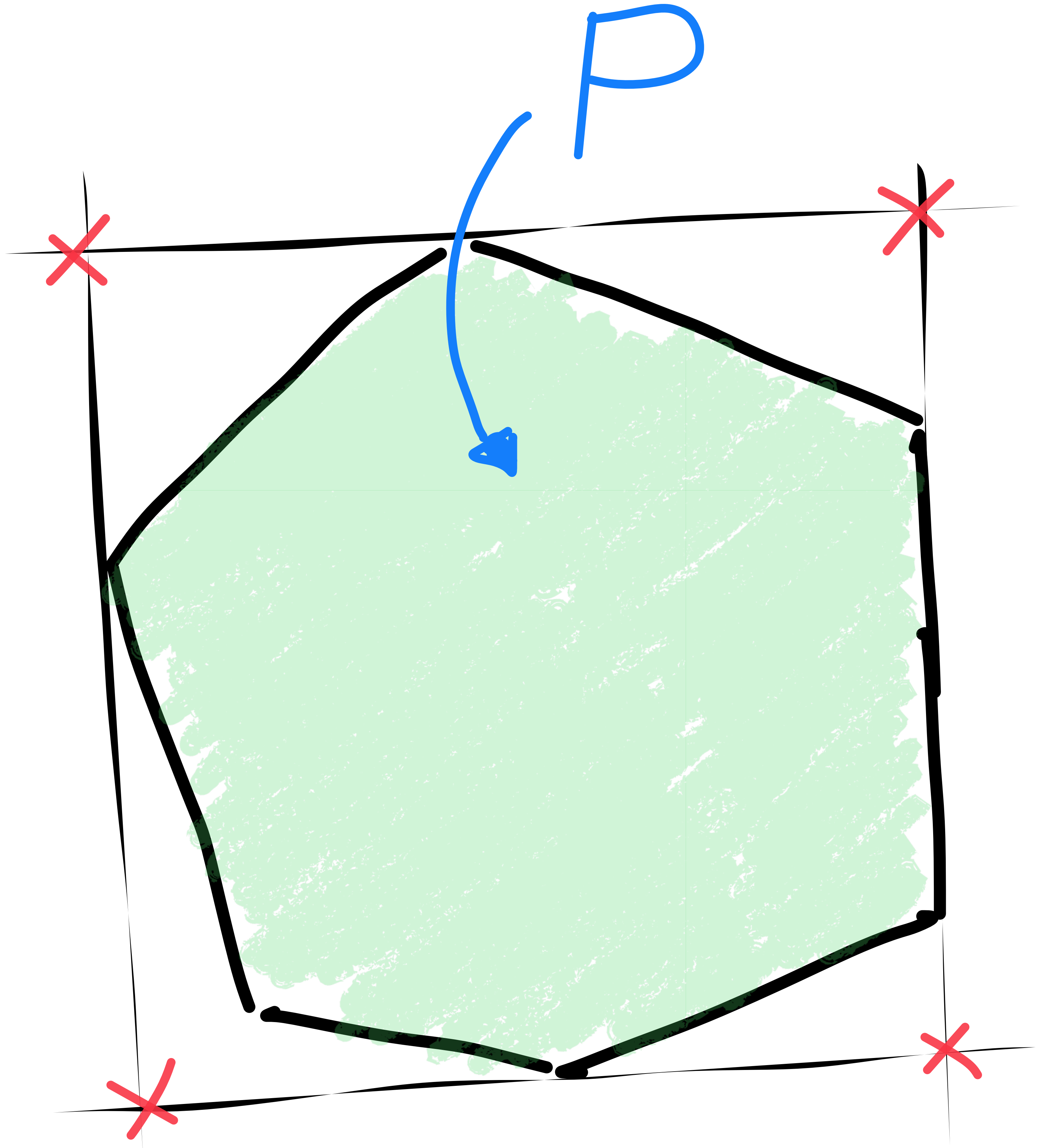
A Classic approach: Chvátal-Gomory Cutting Planes



Chvátal-Gomory Cutting Planes

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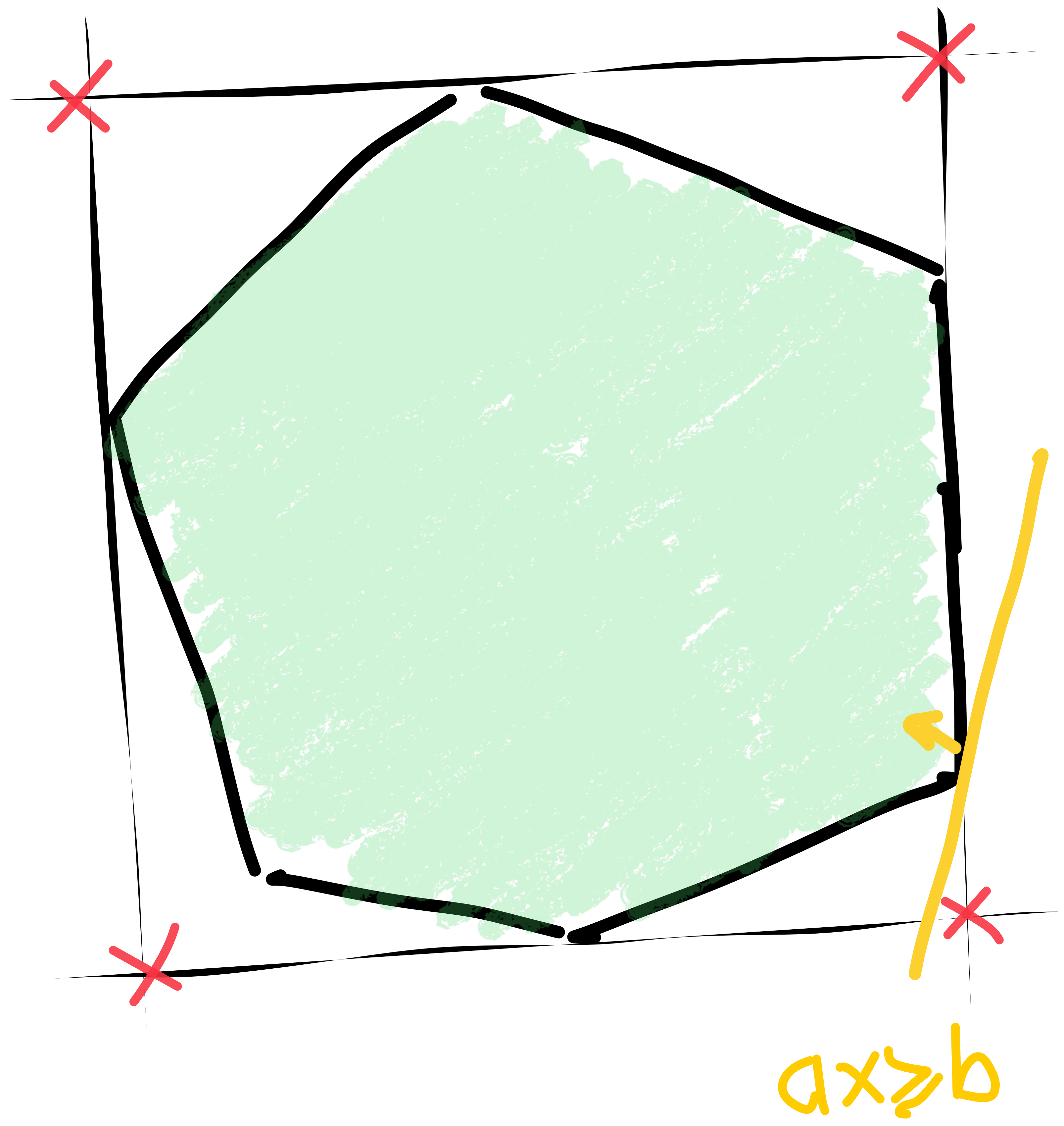

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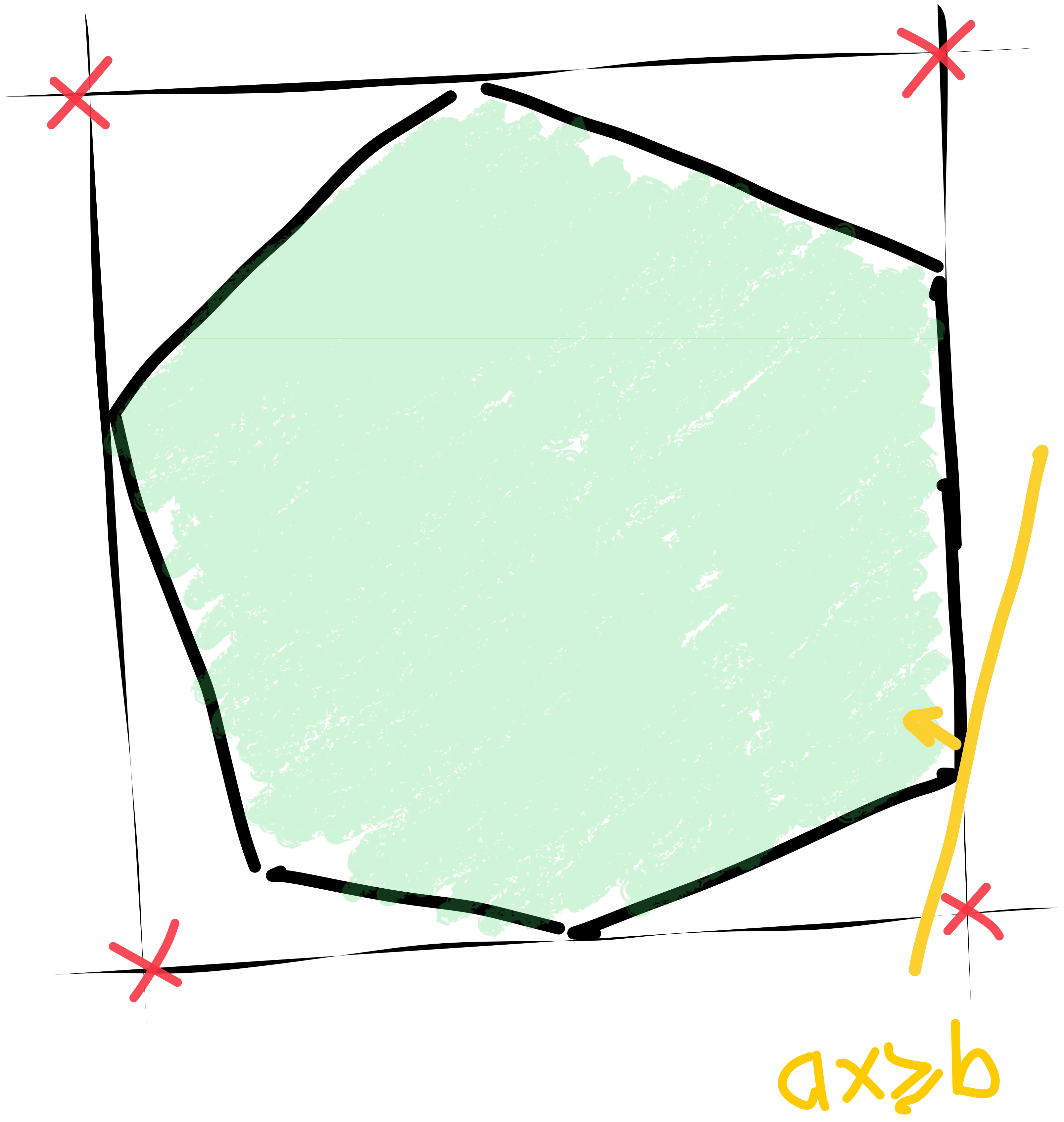

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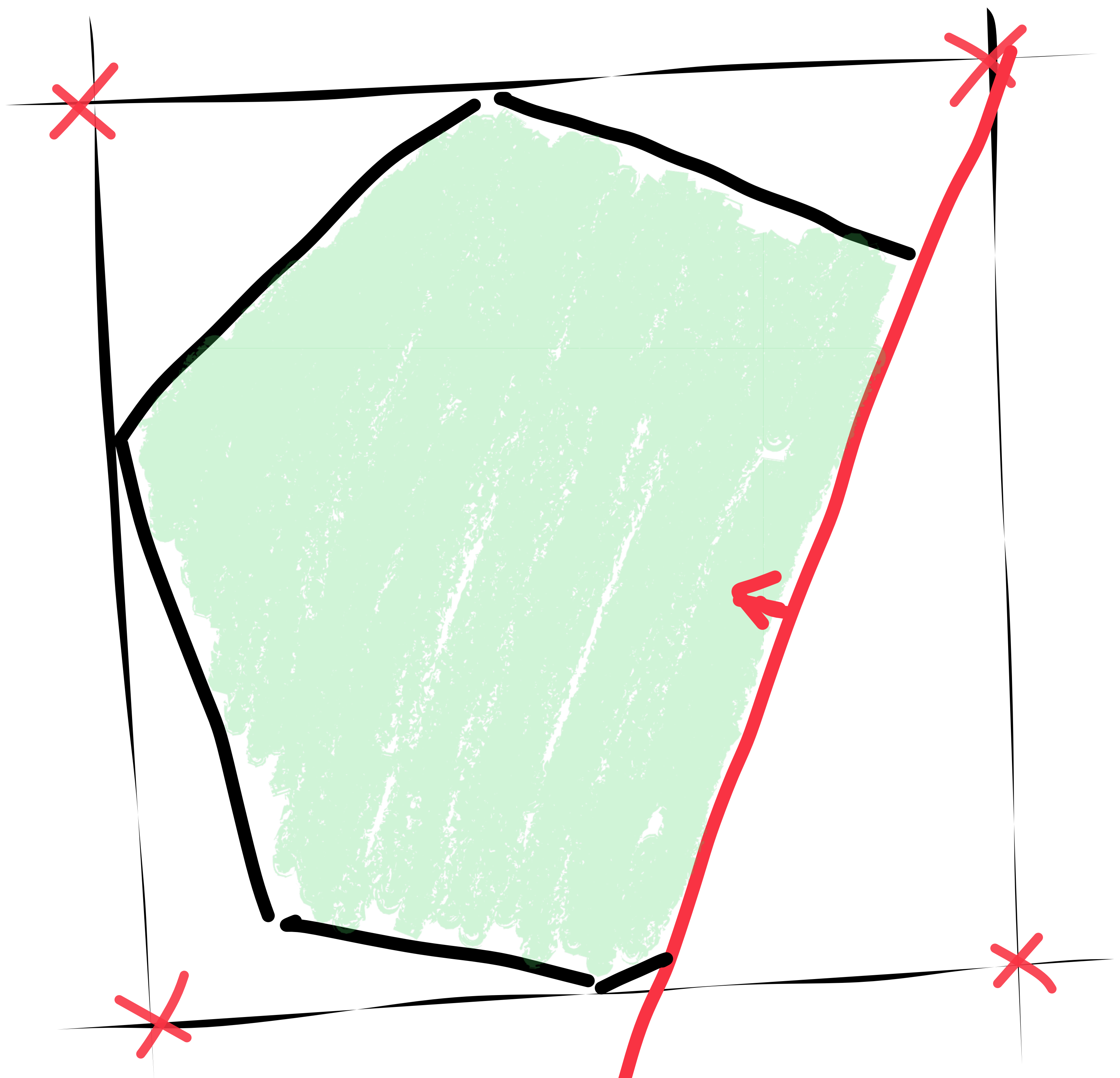

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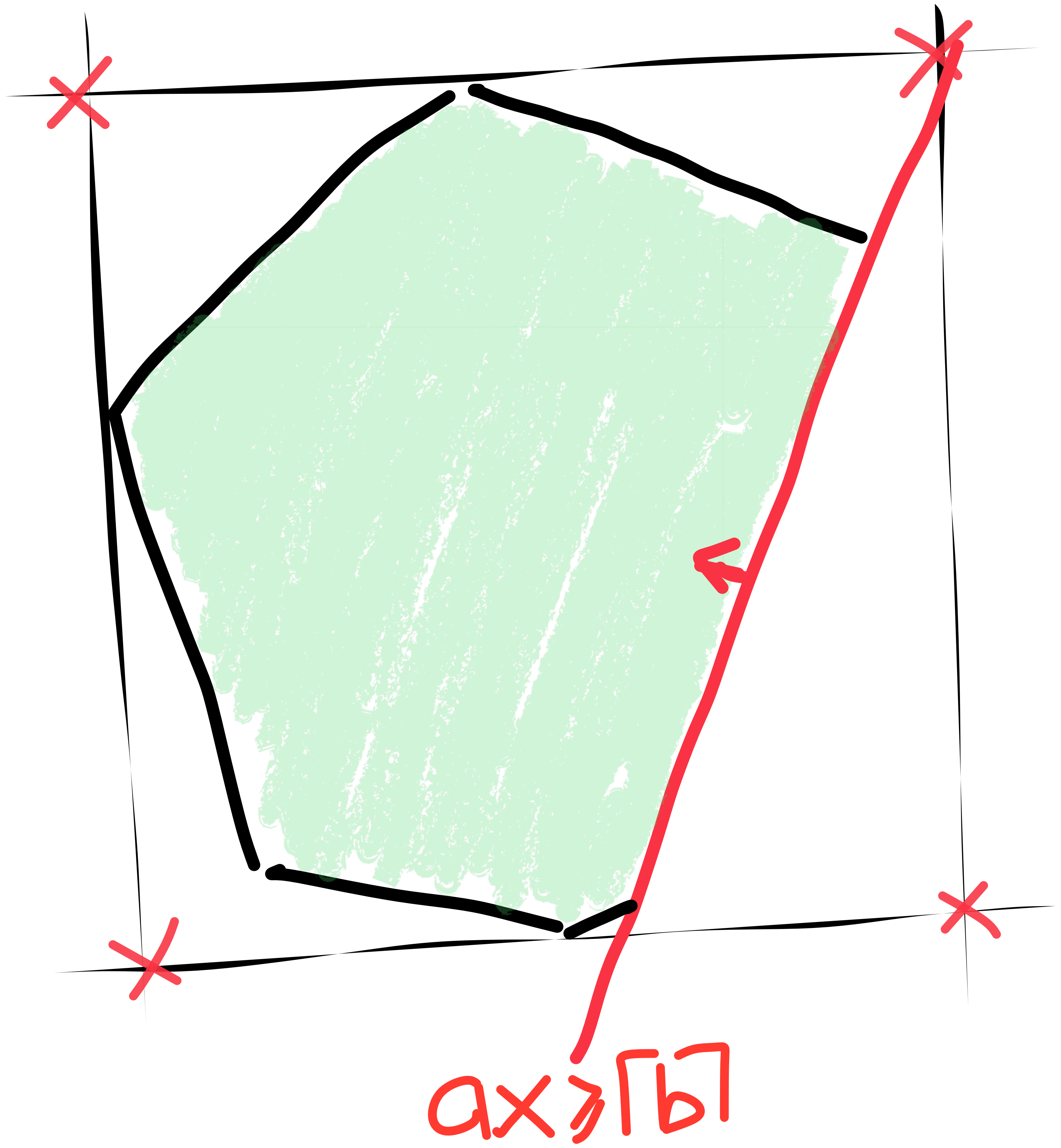

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Heuristically add CG-cuts to P

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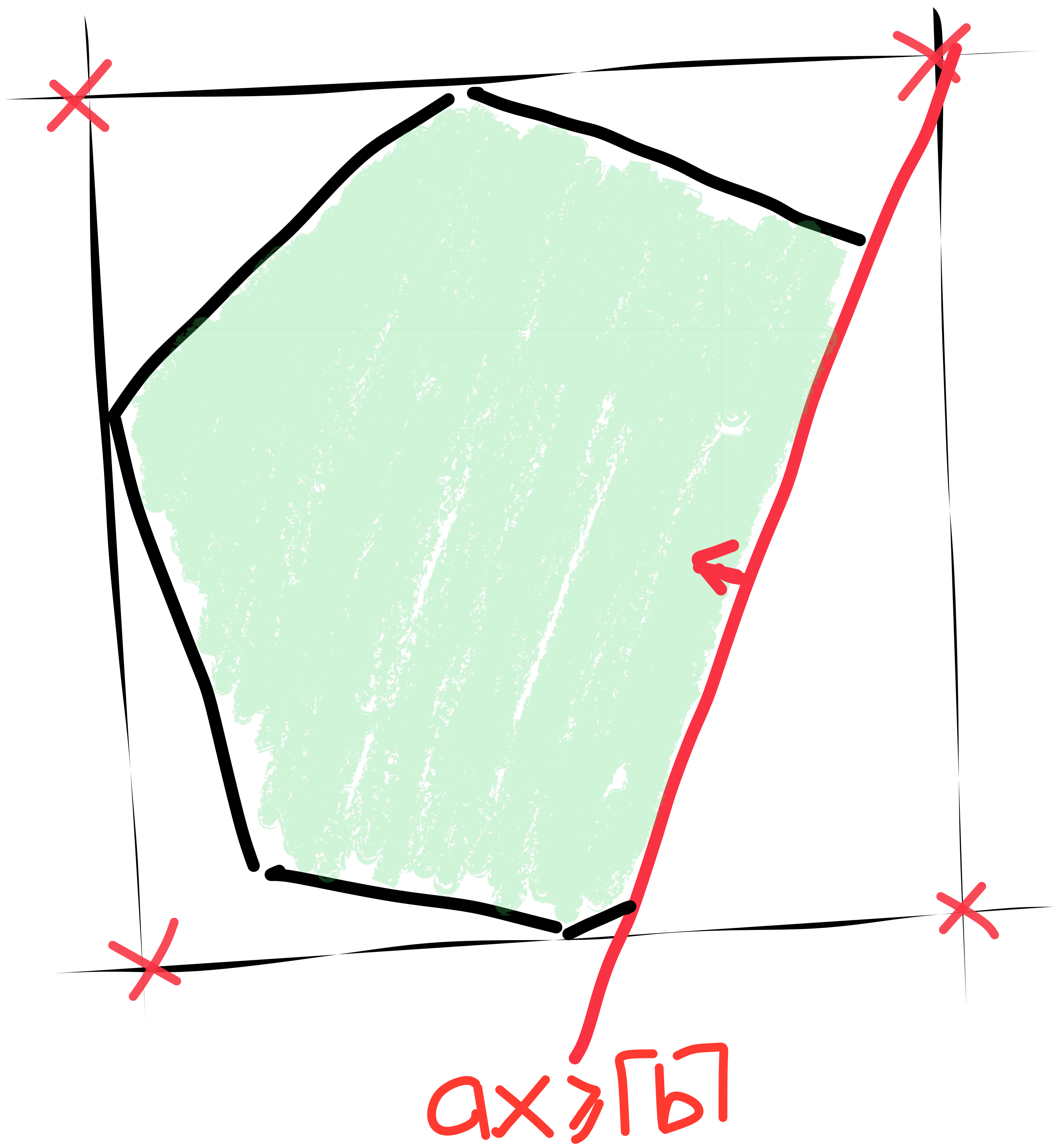

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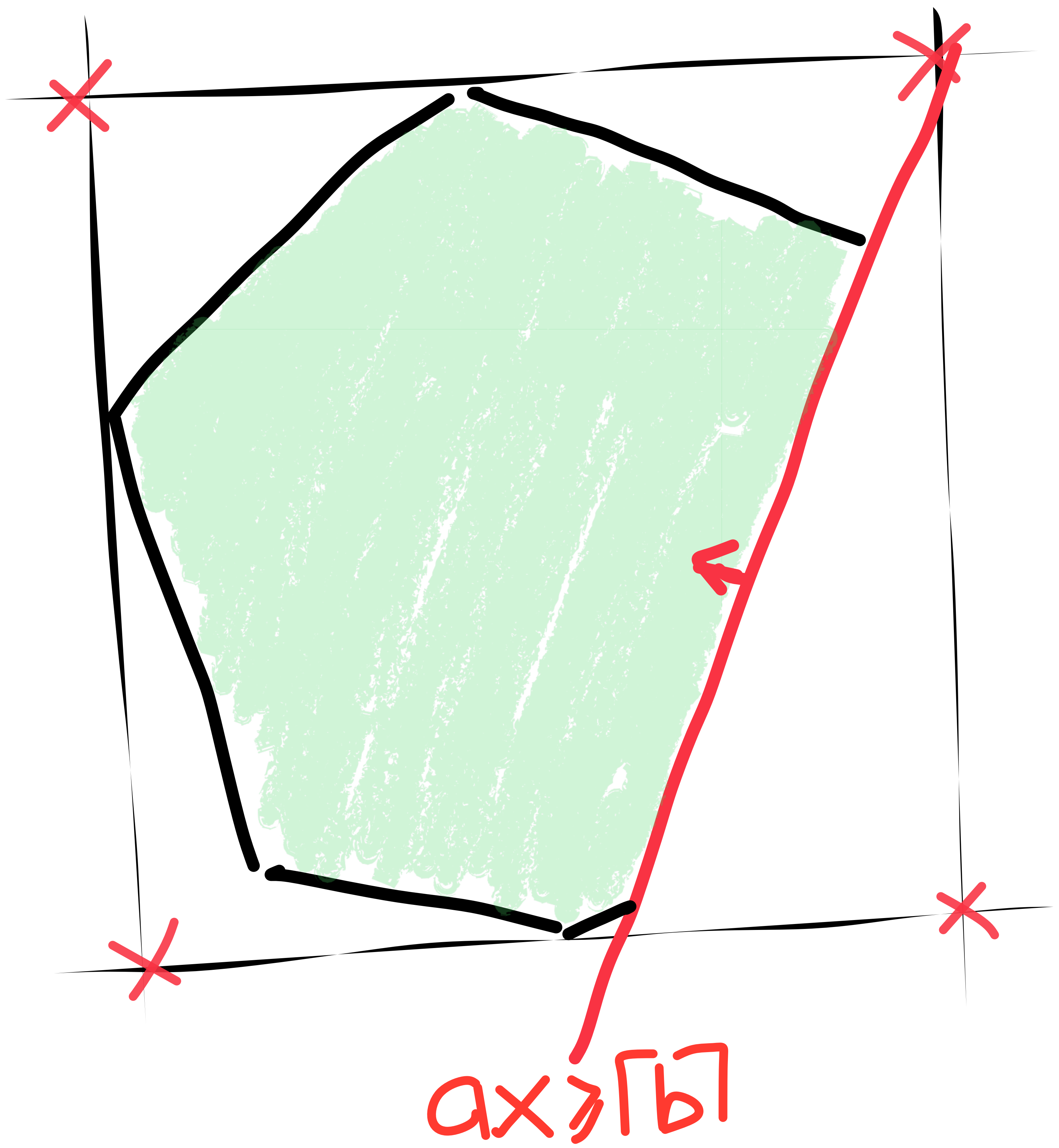

Chvátal-Gomory Cutting Planes

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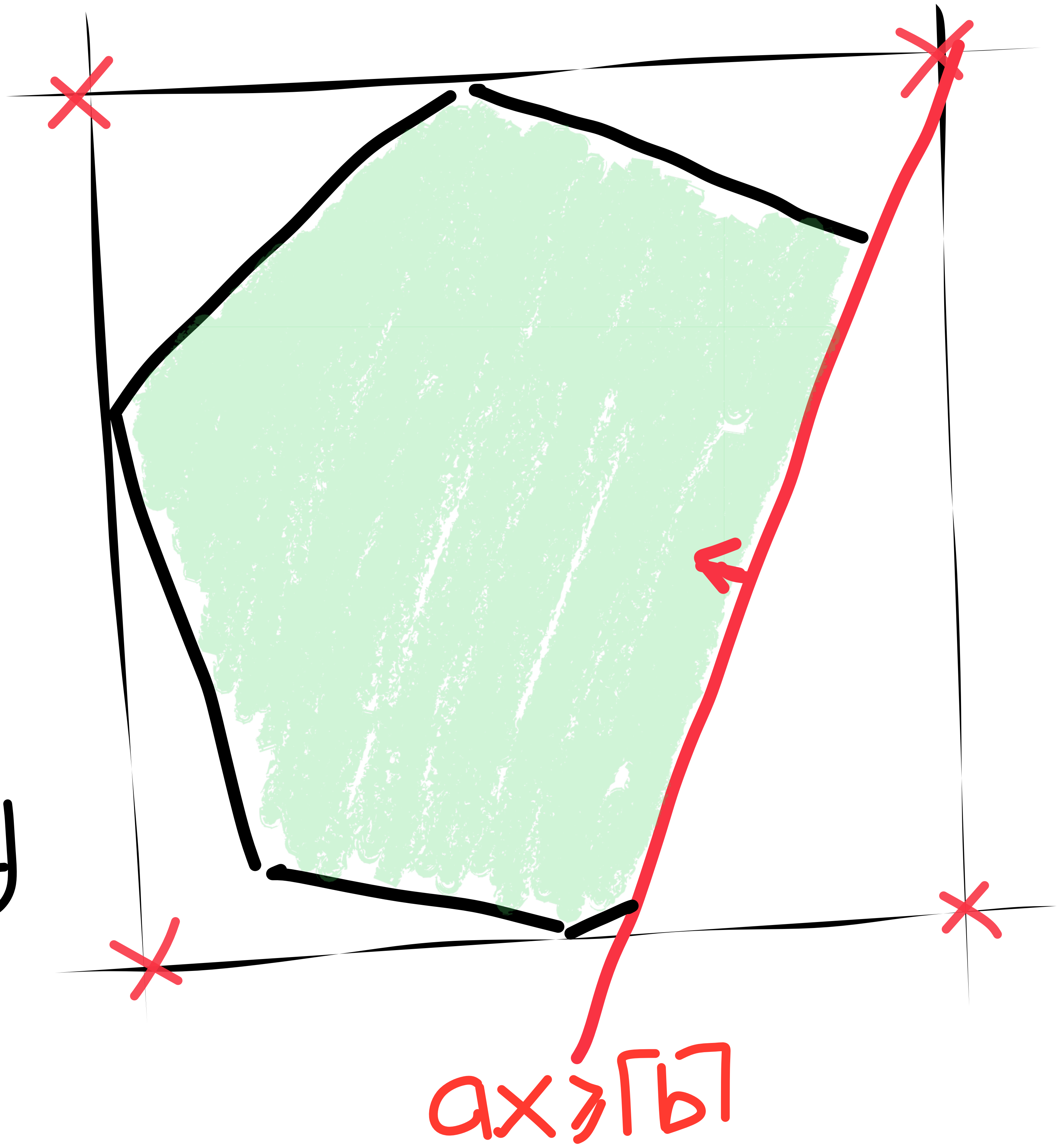
CG-Cutting Planes

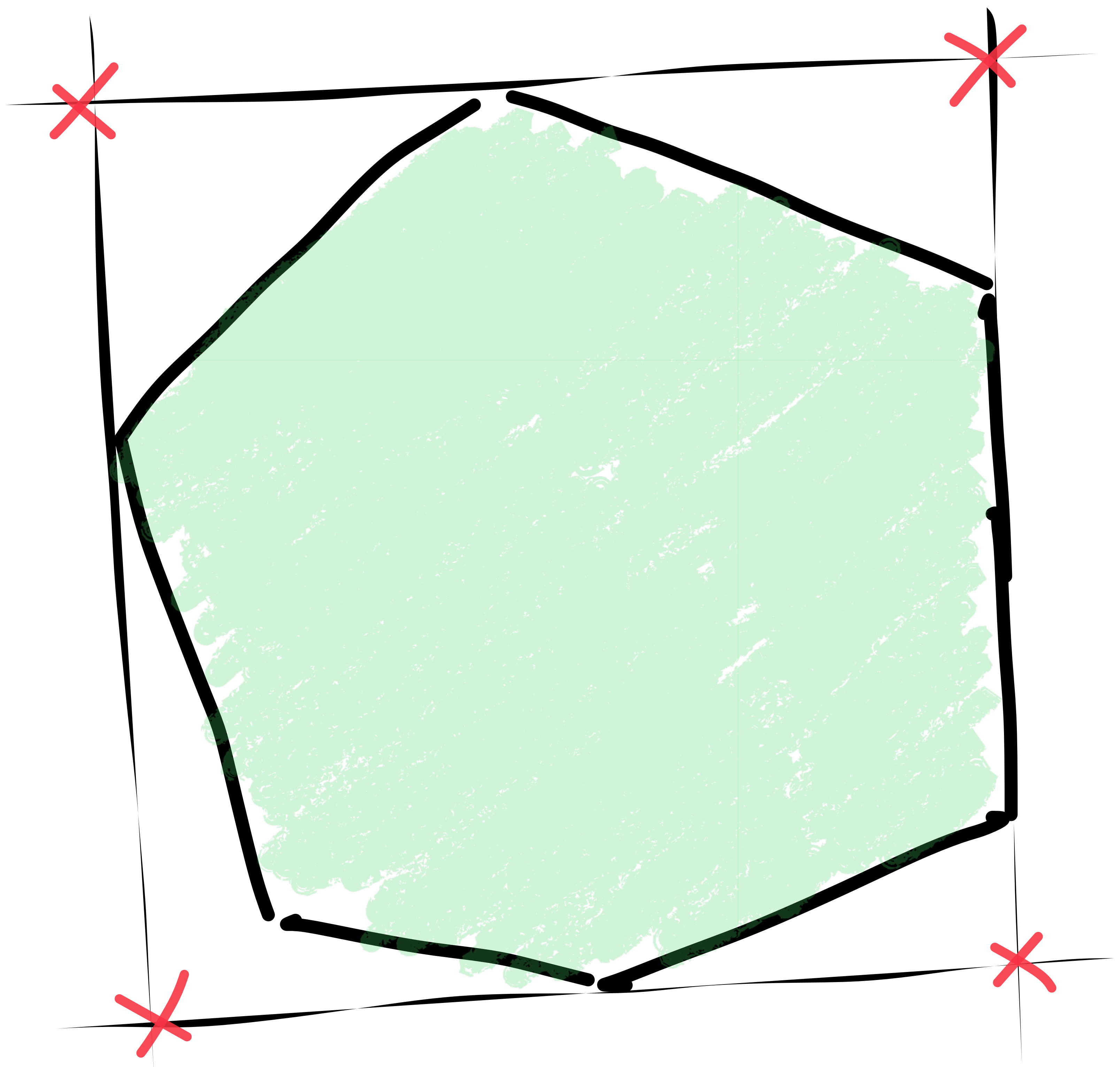
Heuristically add CG-Cuts to P until:

- an integer solution is found
- the empty polytope is deduced



Cutting Planes [CCT87]

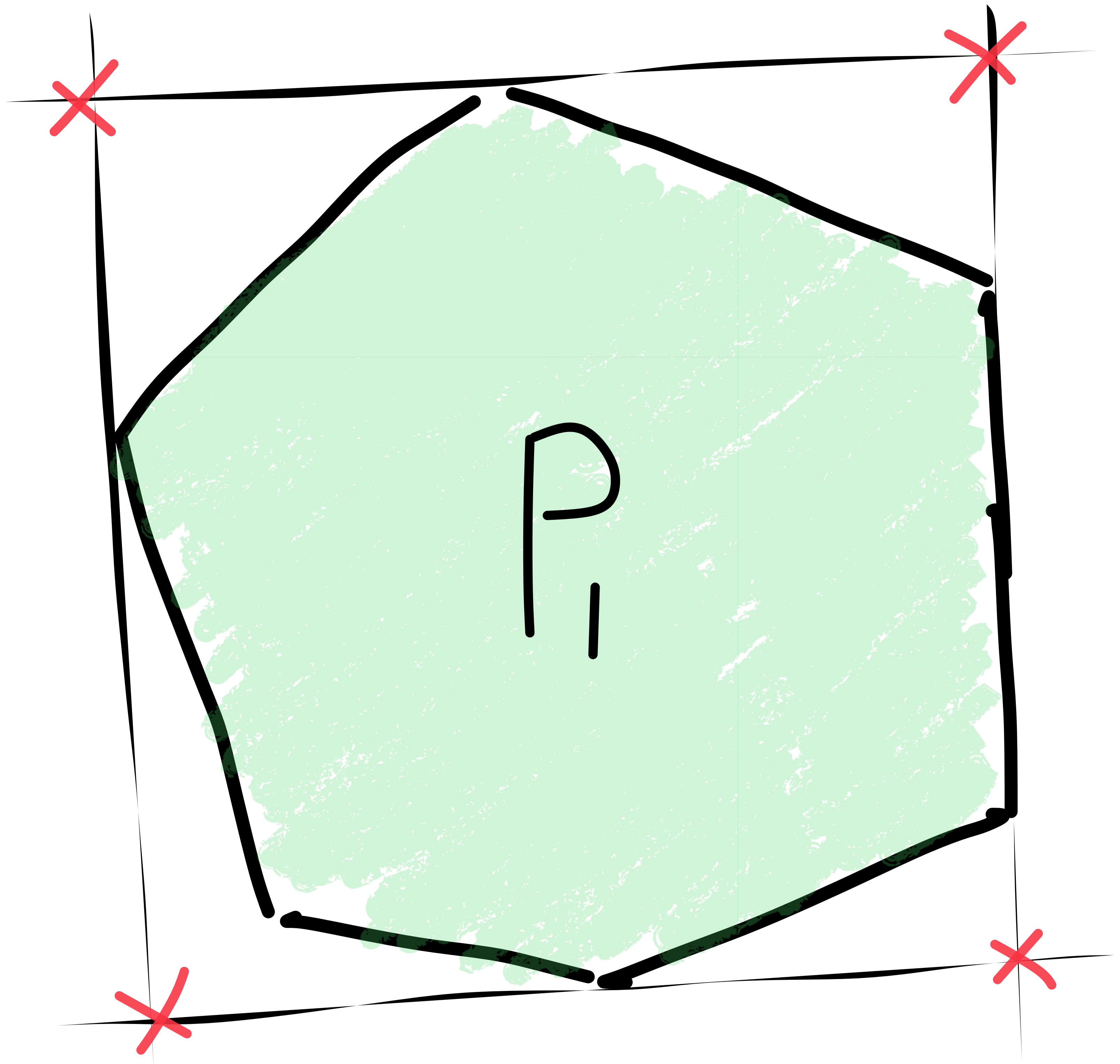
Let $P = \{Ax \geq b\}$ be such that $P \cap \mathbb{Z}^n = \emptyset$



Cutting Planes

Let $P = \{Ax \geq b\}$ be such that $P \cap \mathbb{Z}^n = \emptyset$

A CP proof that $P \cap \mathbb{Z}^n = \emptyset$ is a sequence of polytopes $P = P_1, \ldots, P_s = \emptyset$ s.t. P_{i+1} is deduced from P_i by a CG-cut



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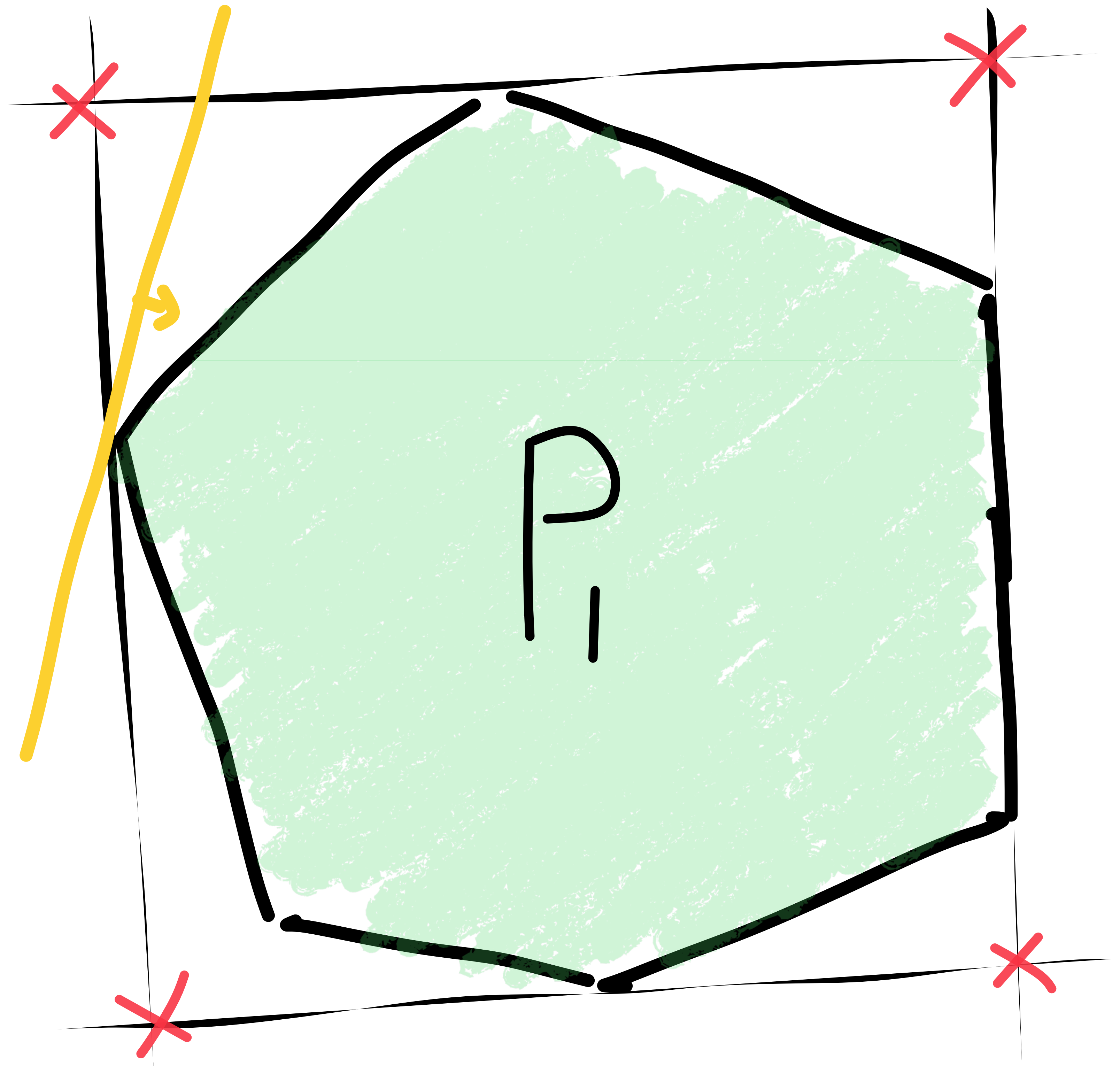

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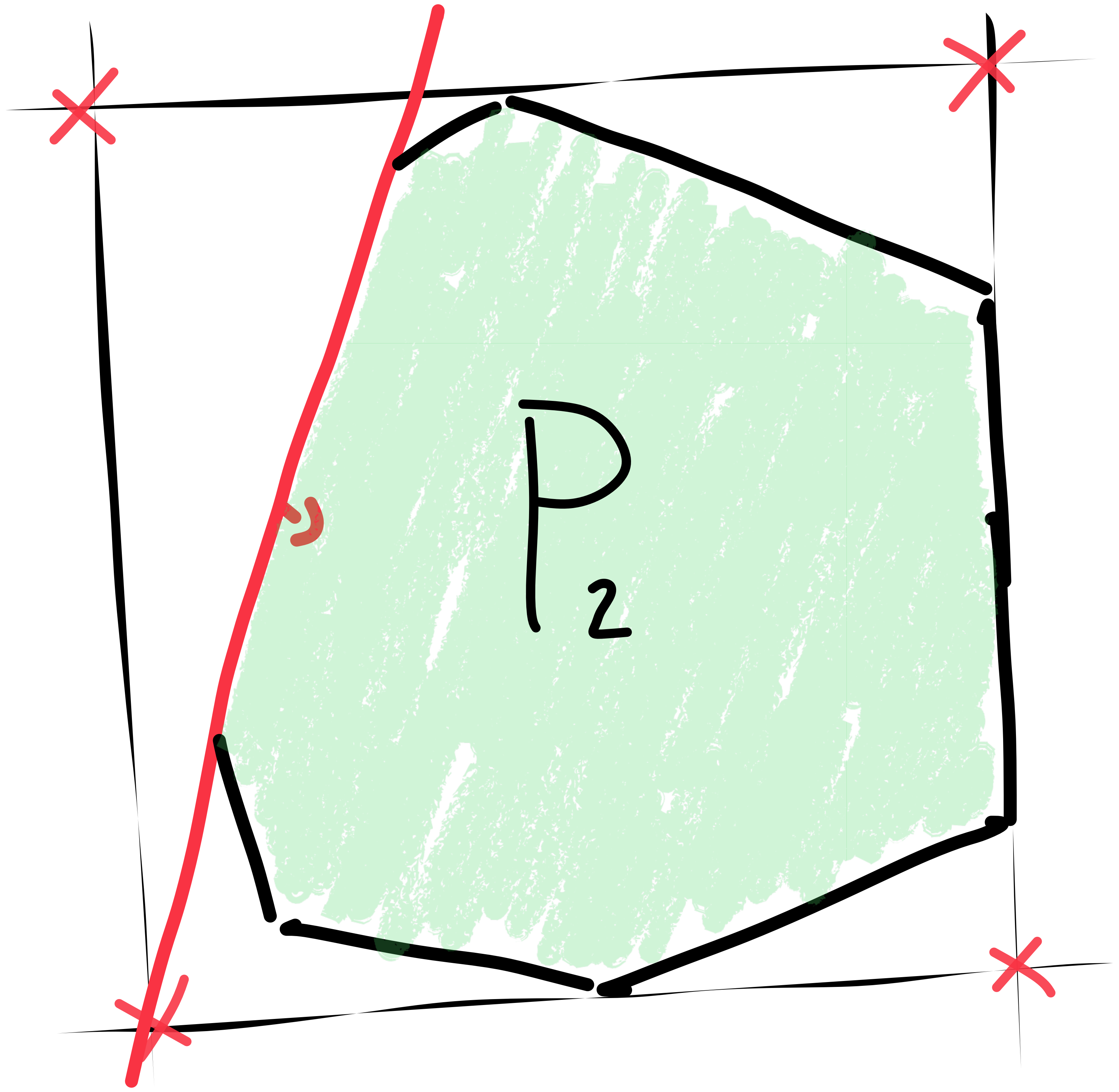

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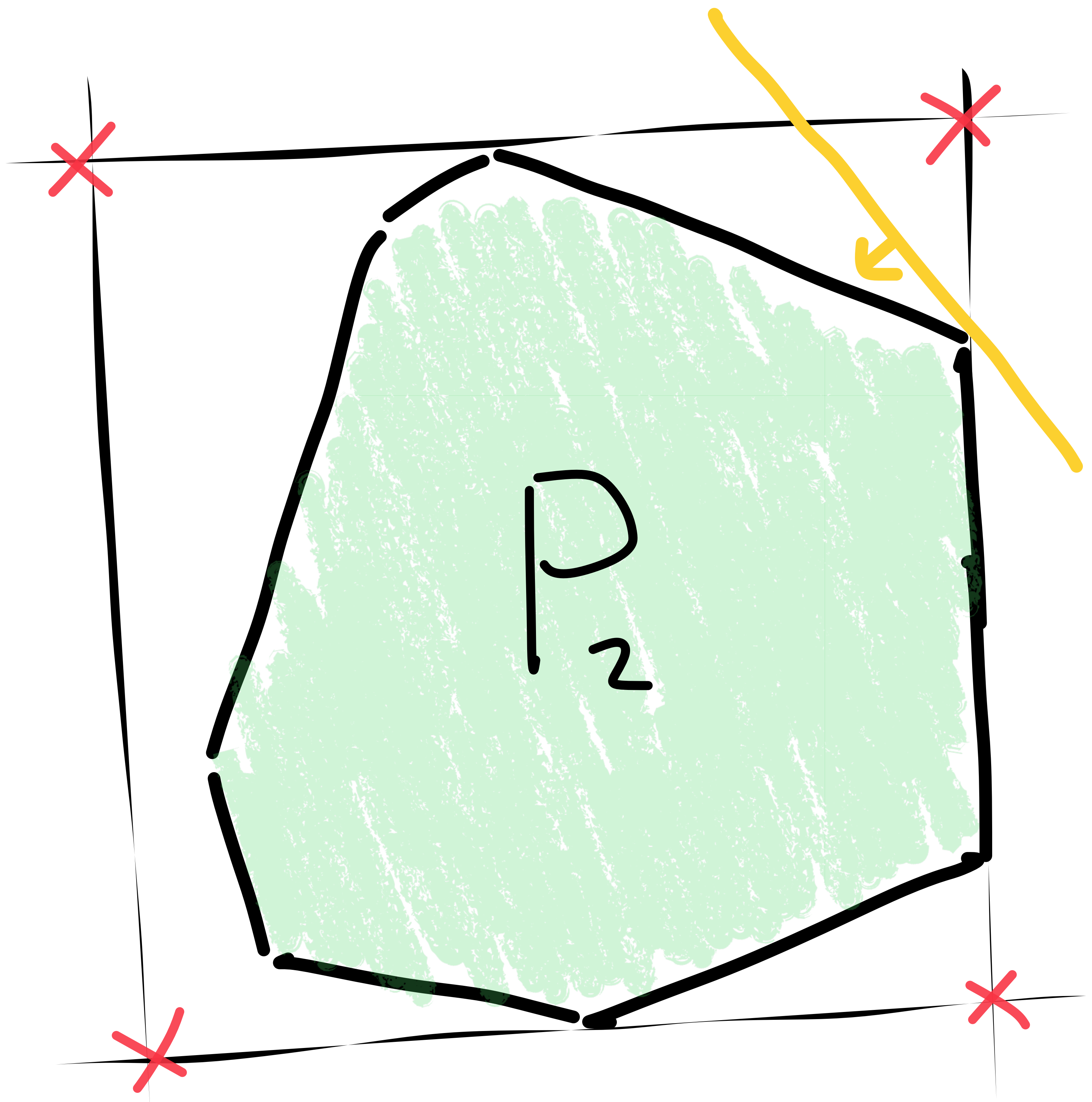

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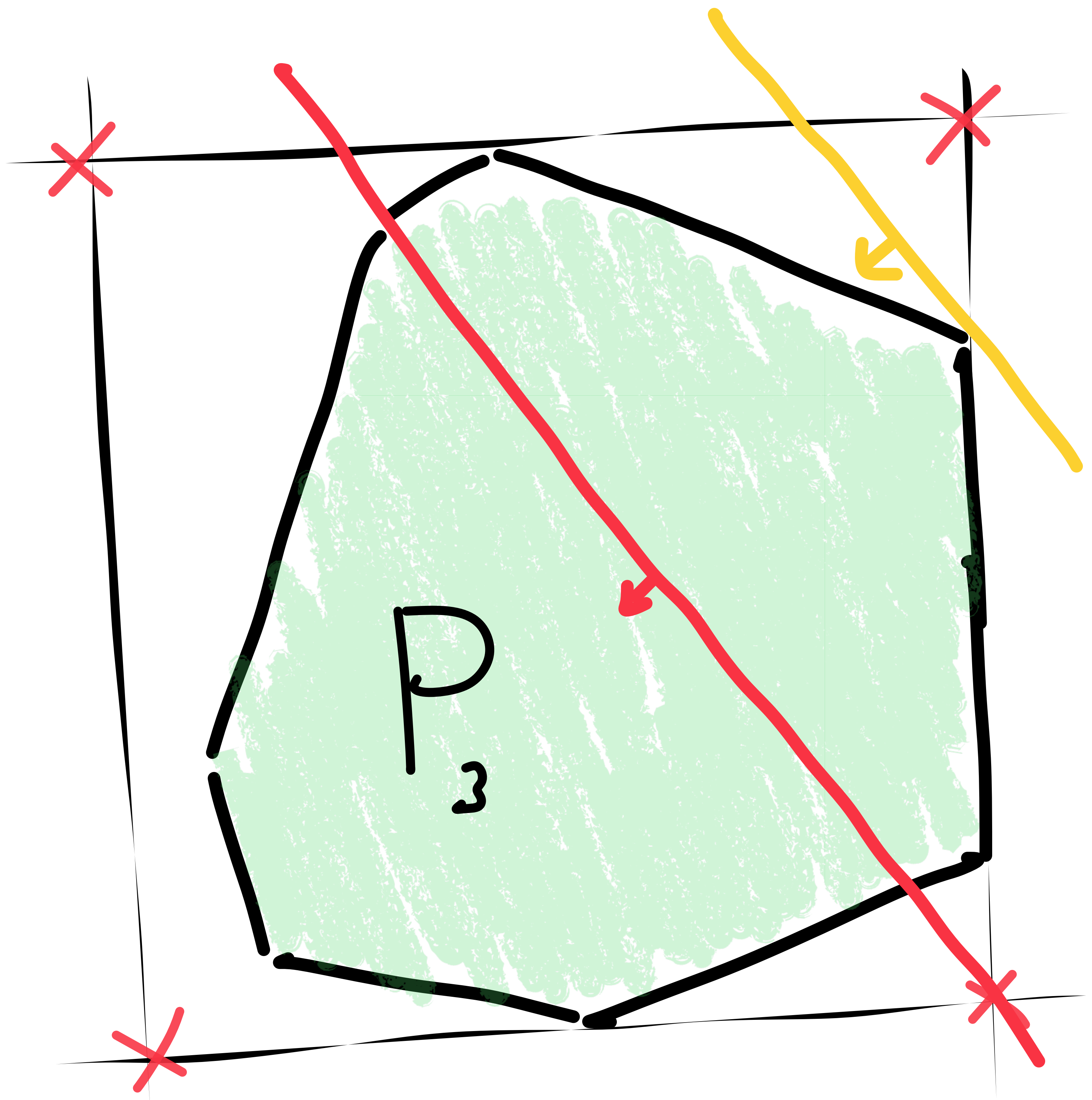

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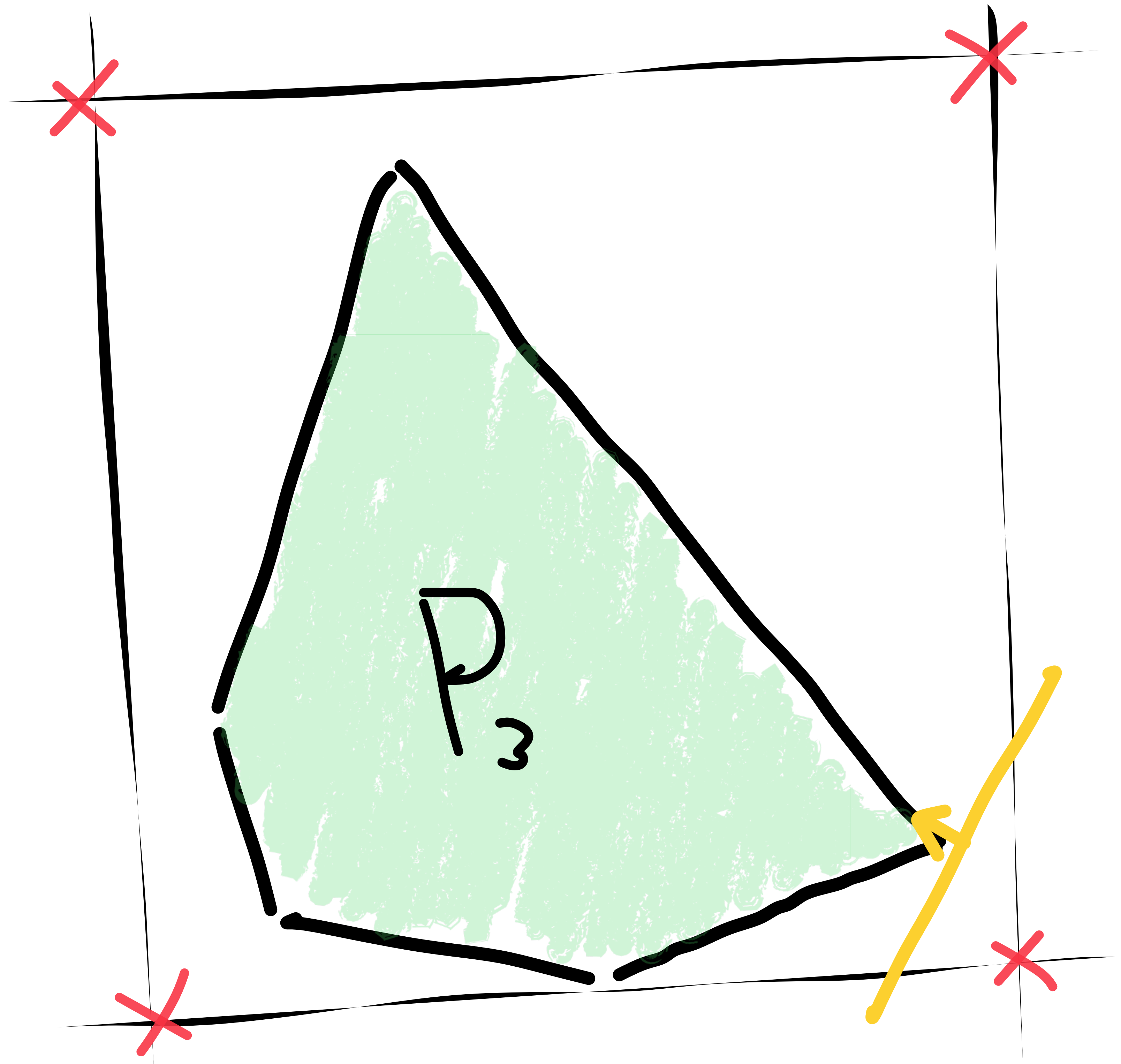

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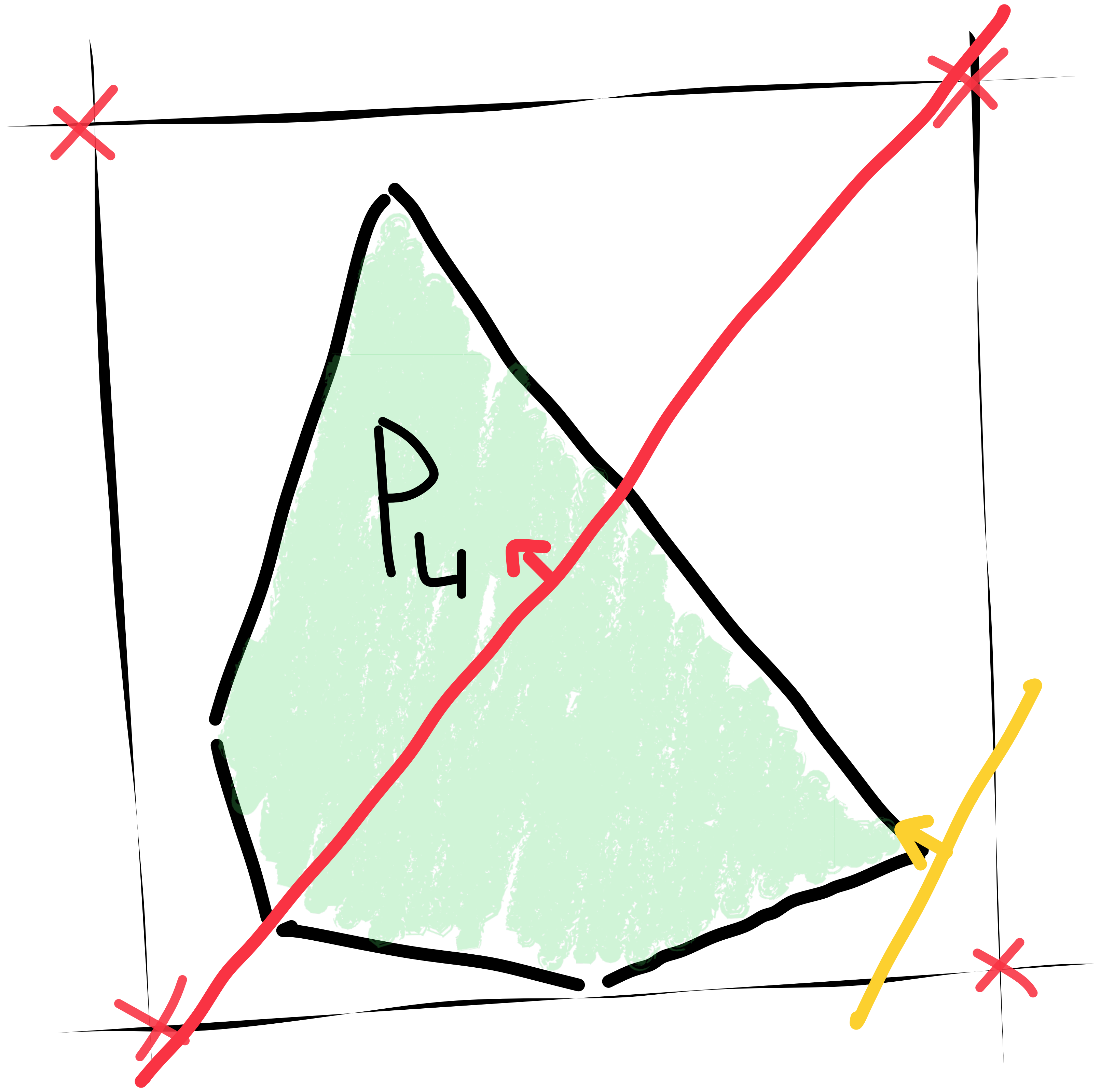

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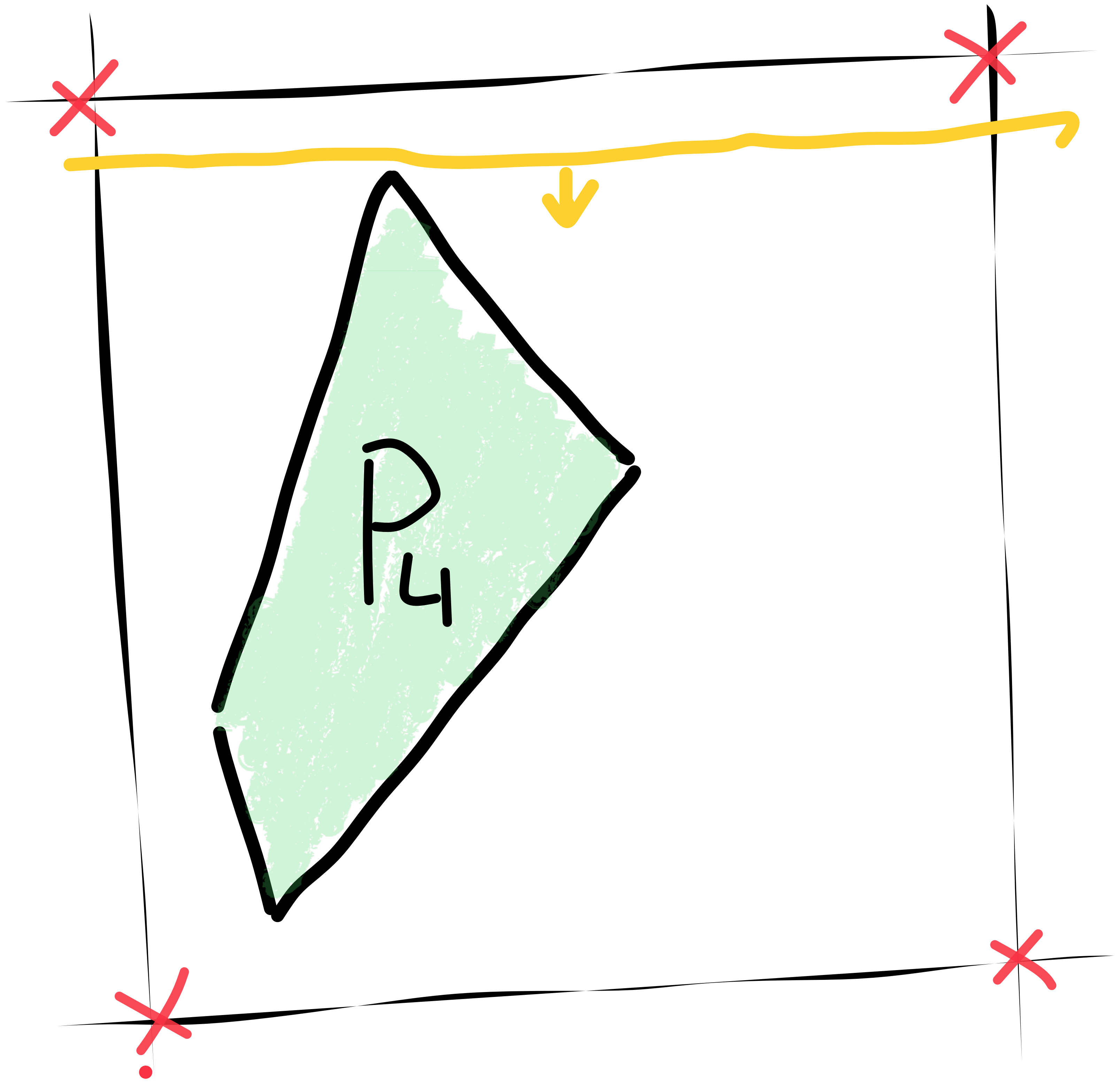

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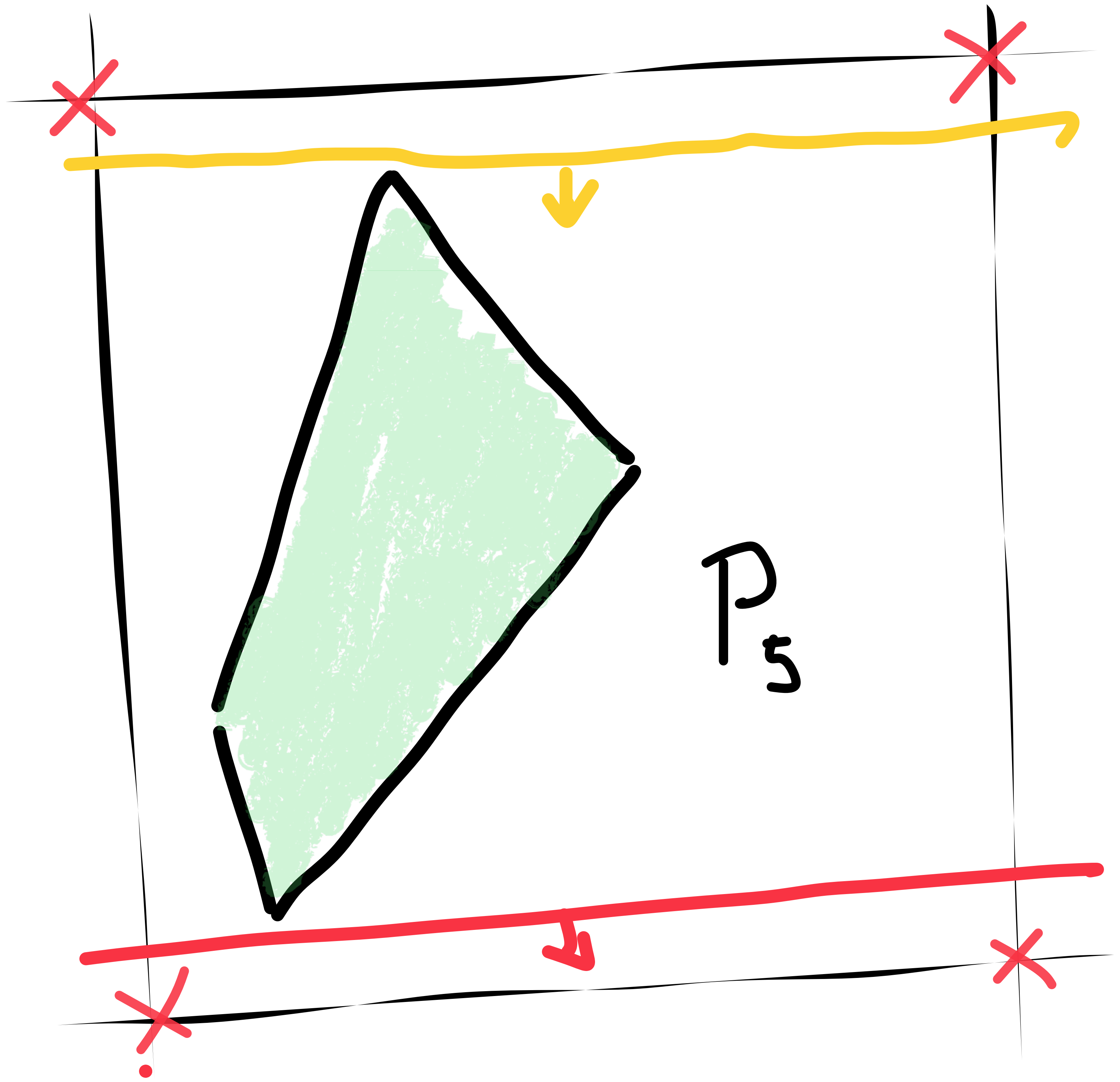

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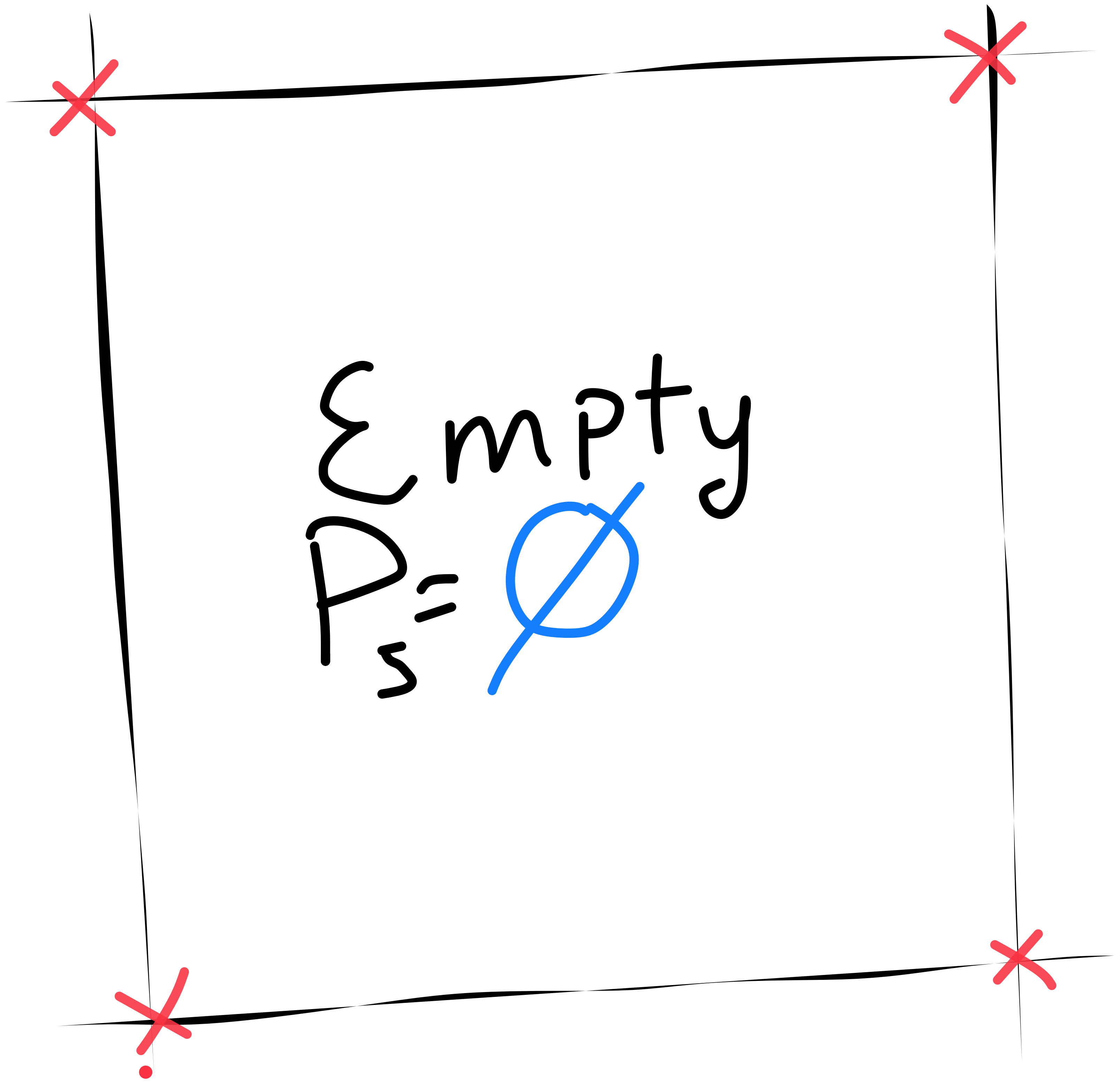

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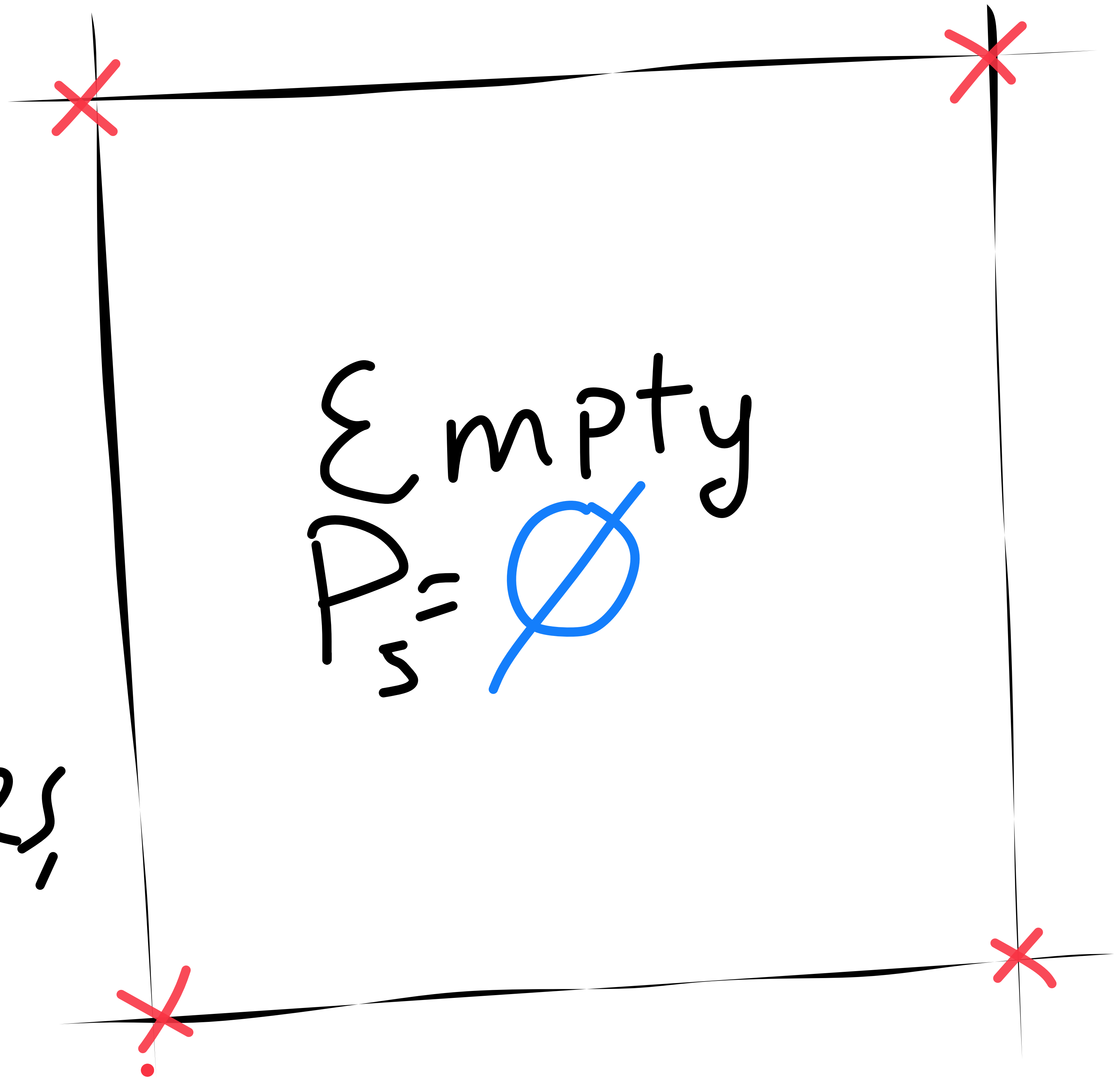
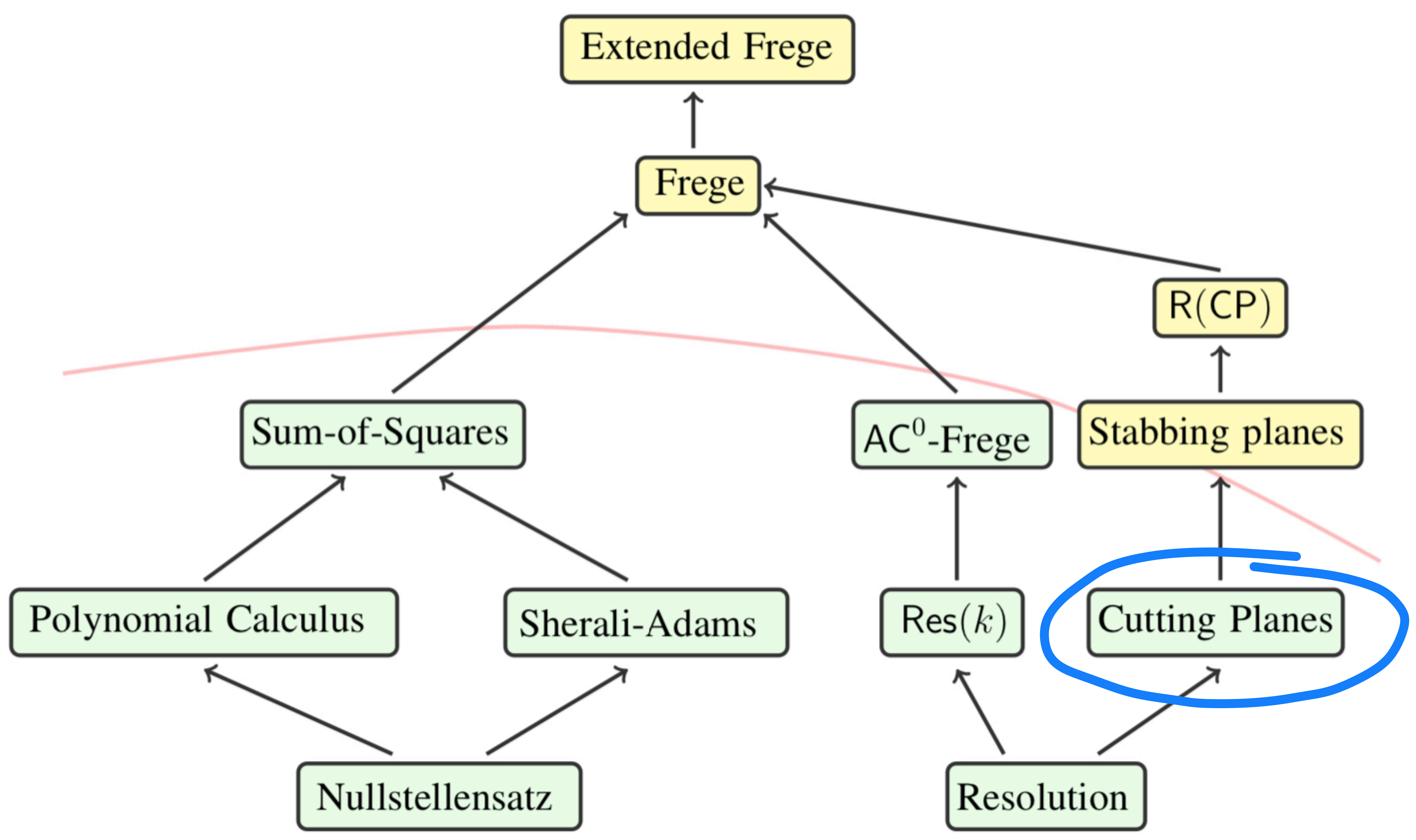


Proof Size: the number of polytopes, s

Cutting Planes

Powerful & Well-studied algebraic proof system



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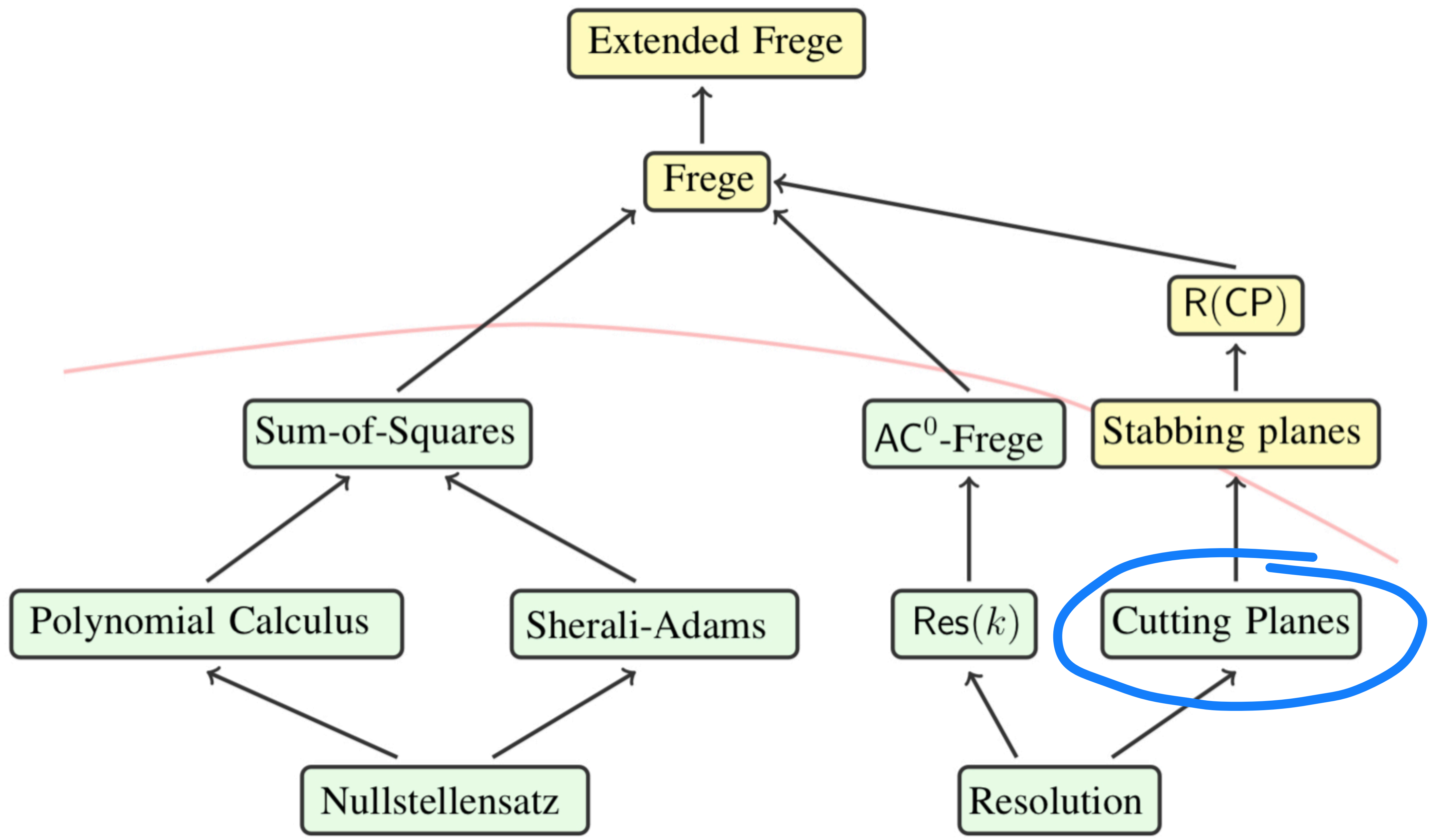
▷ Short proofs of PHP



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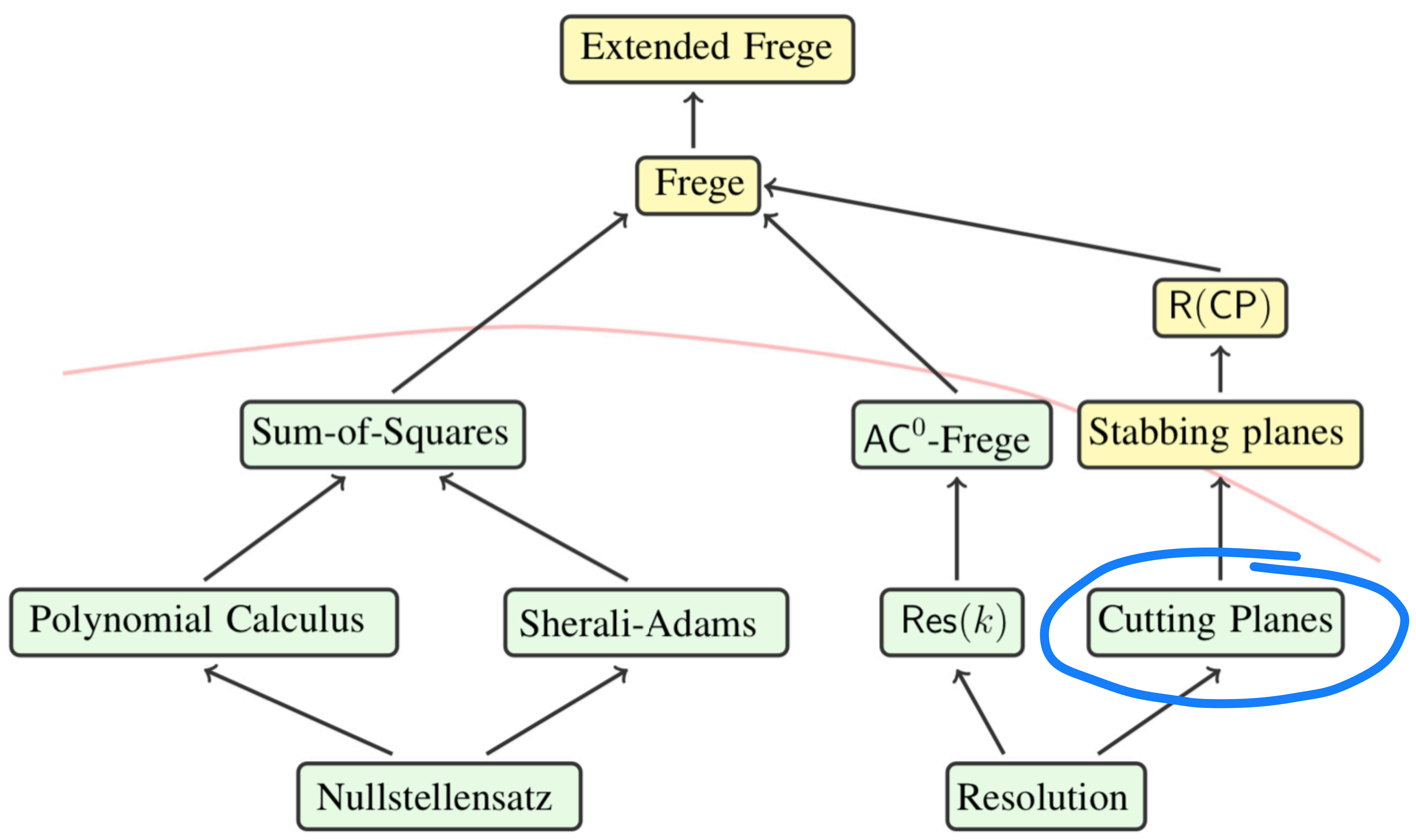
- ▷ Short proofs of PHP
- ▷ Exponential lowerbounds [Pud97]



Integer Programming

Modern IP Algorithms combine cutting planes with a branch-and-bound procedure

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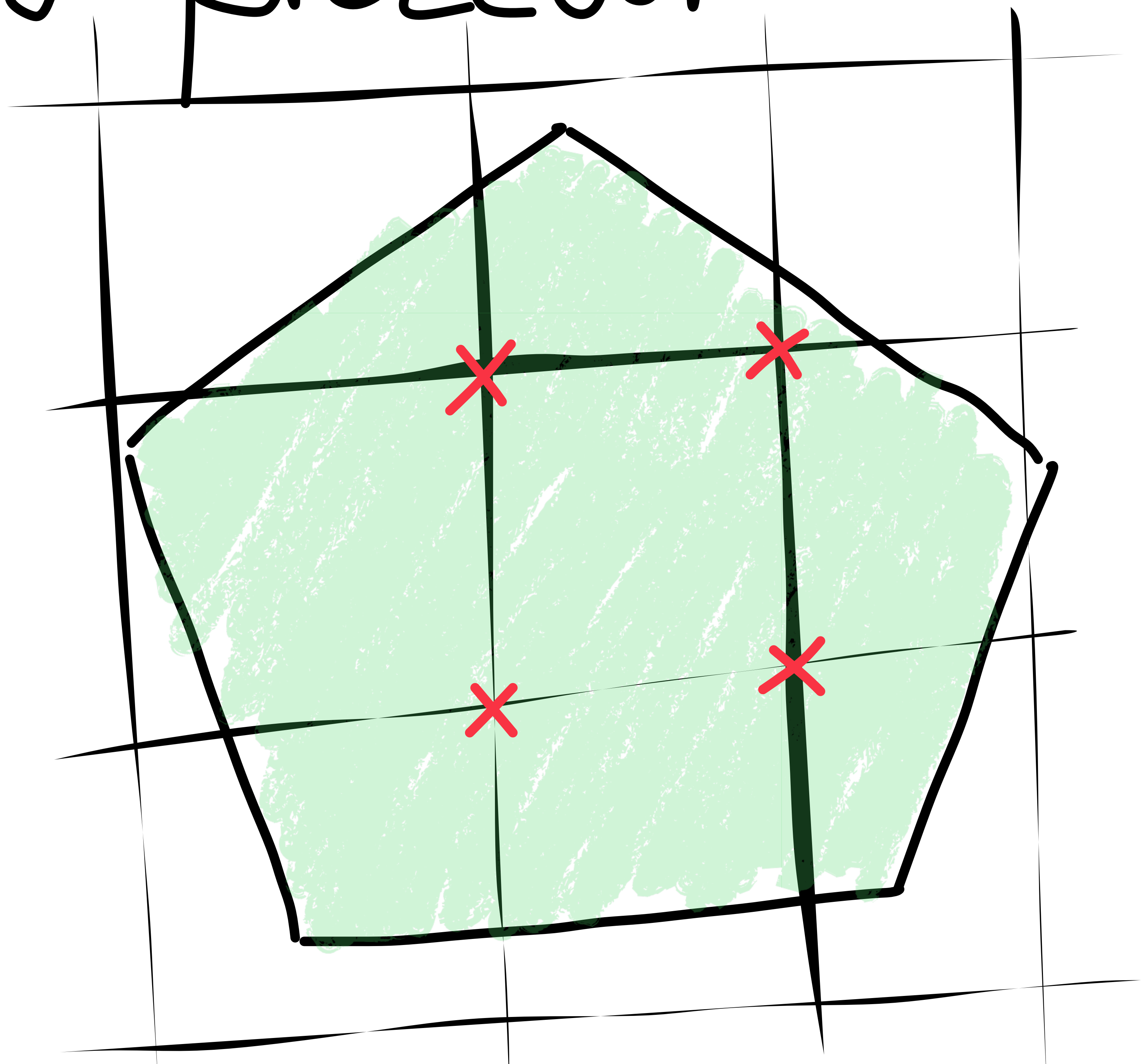
<u>Branch</u>-<u>and</u>-<u>Cut</u> On input P,



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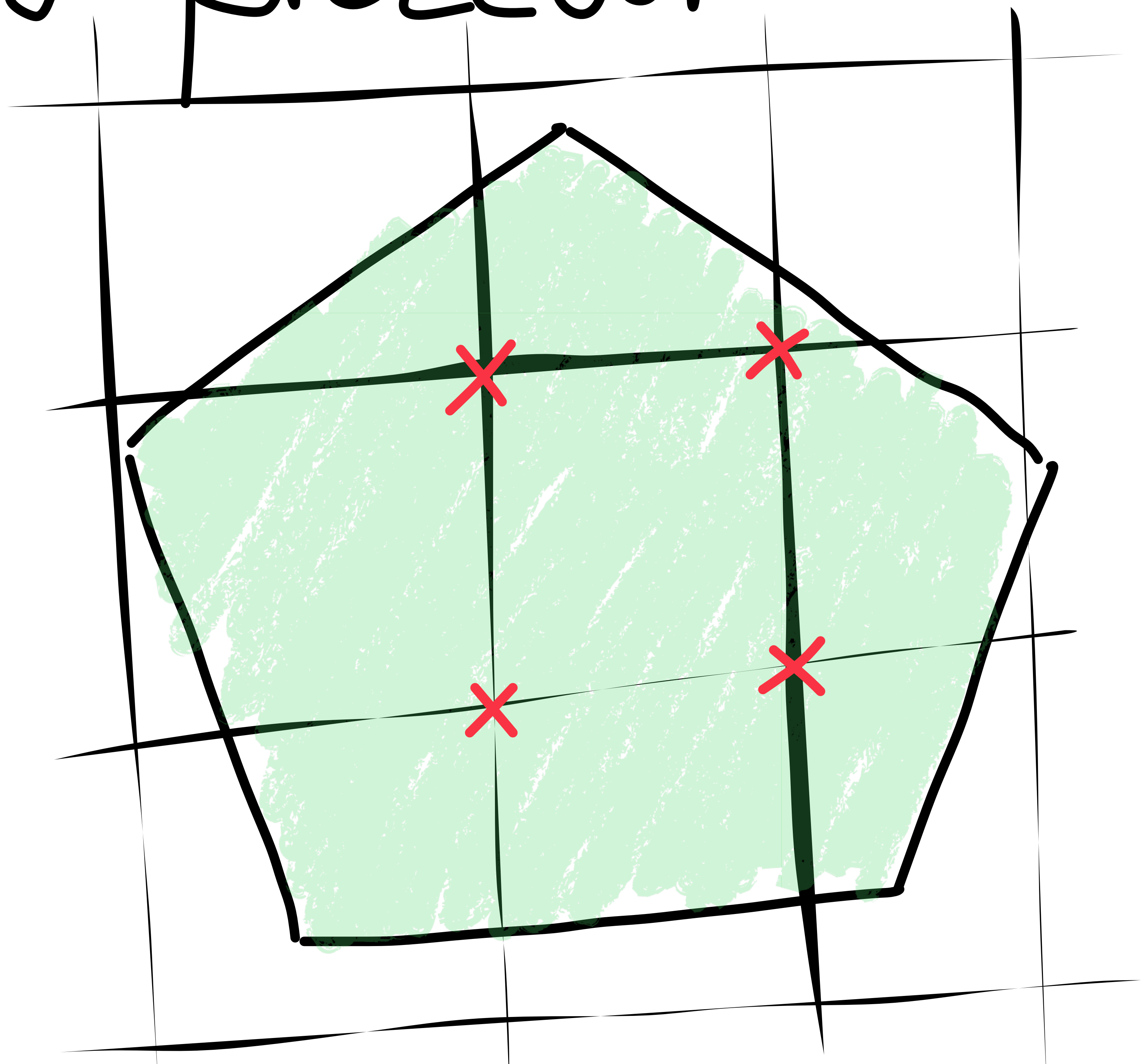

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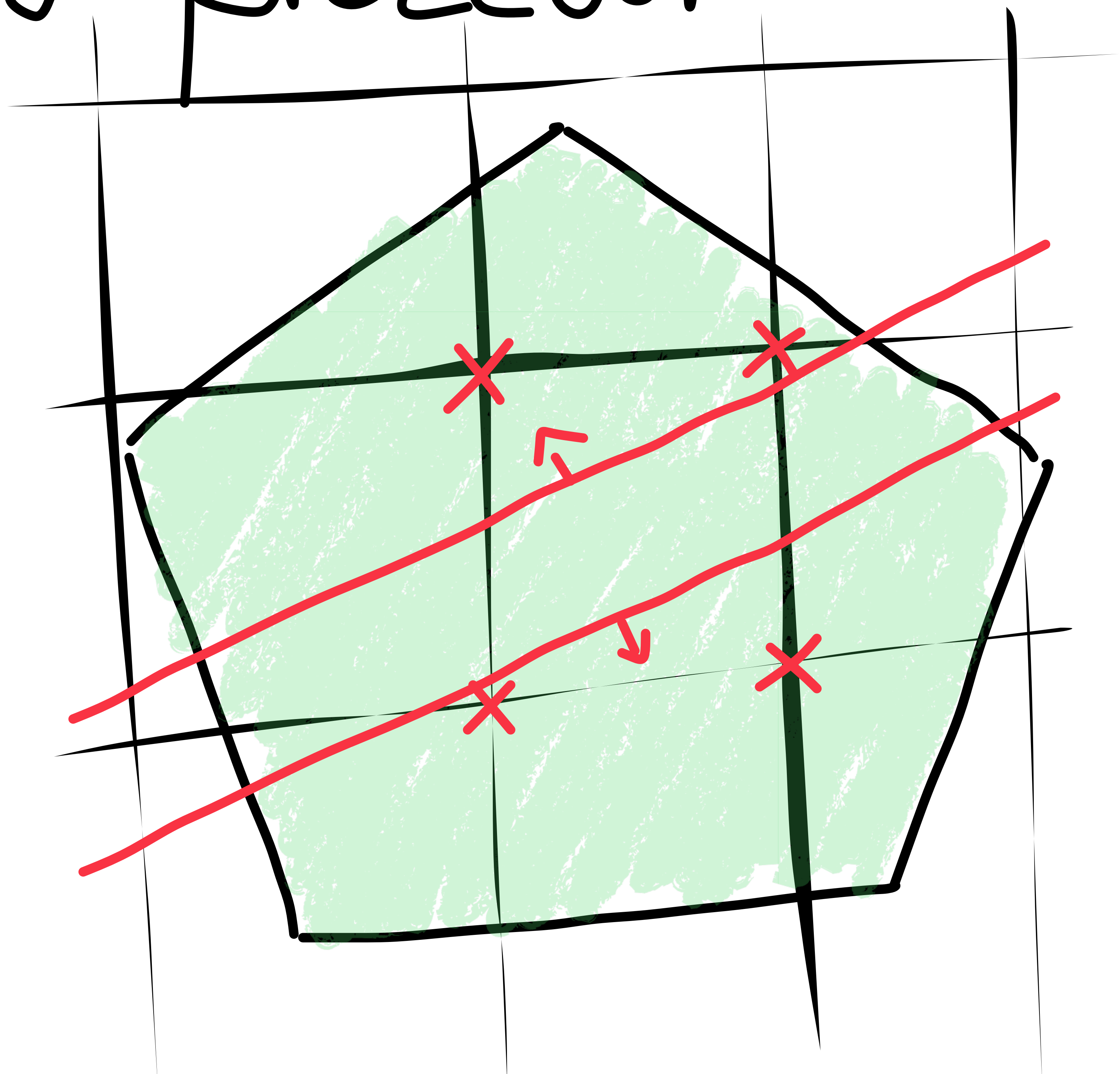

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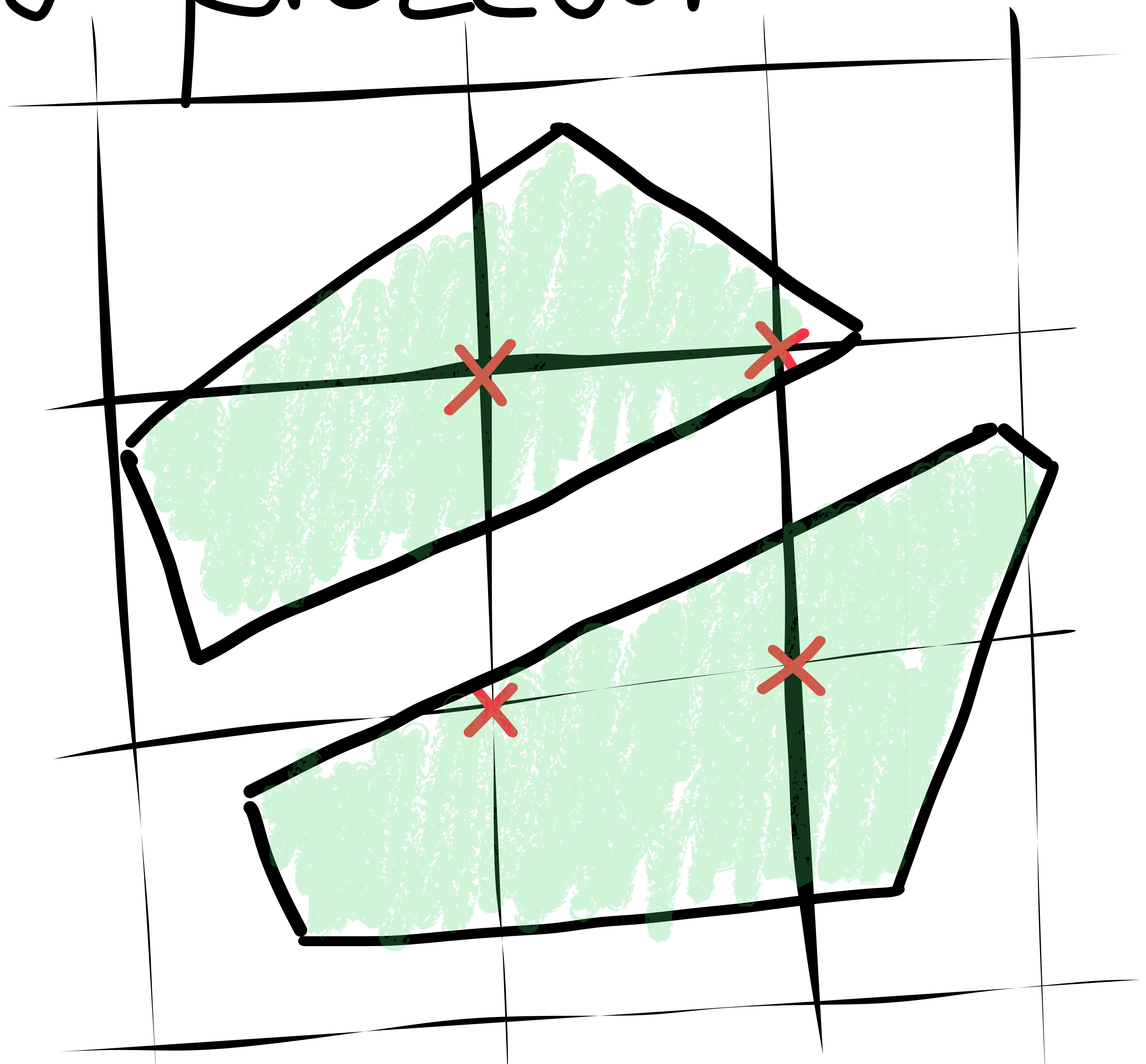

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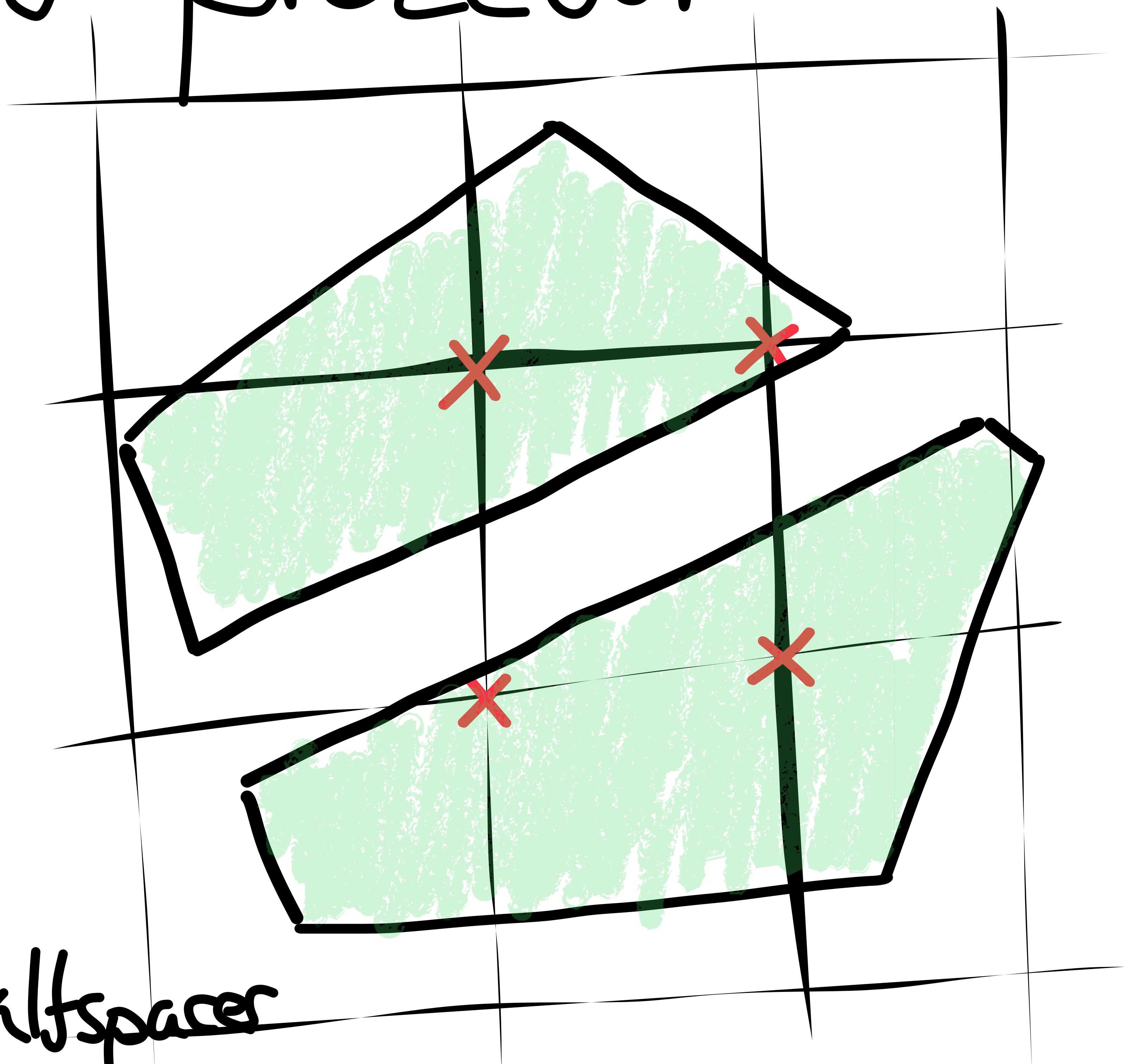

▷ In practice, P is broken into $P \cap \{x : ax \geq b\}$ and $P \cap \{x : ax \leq b-1\}$ for some class of halfspaces

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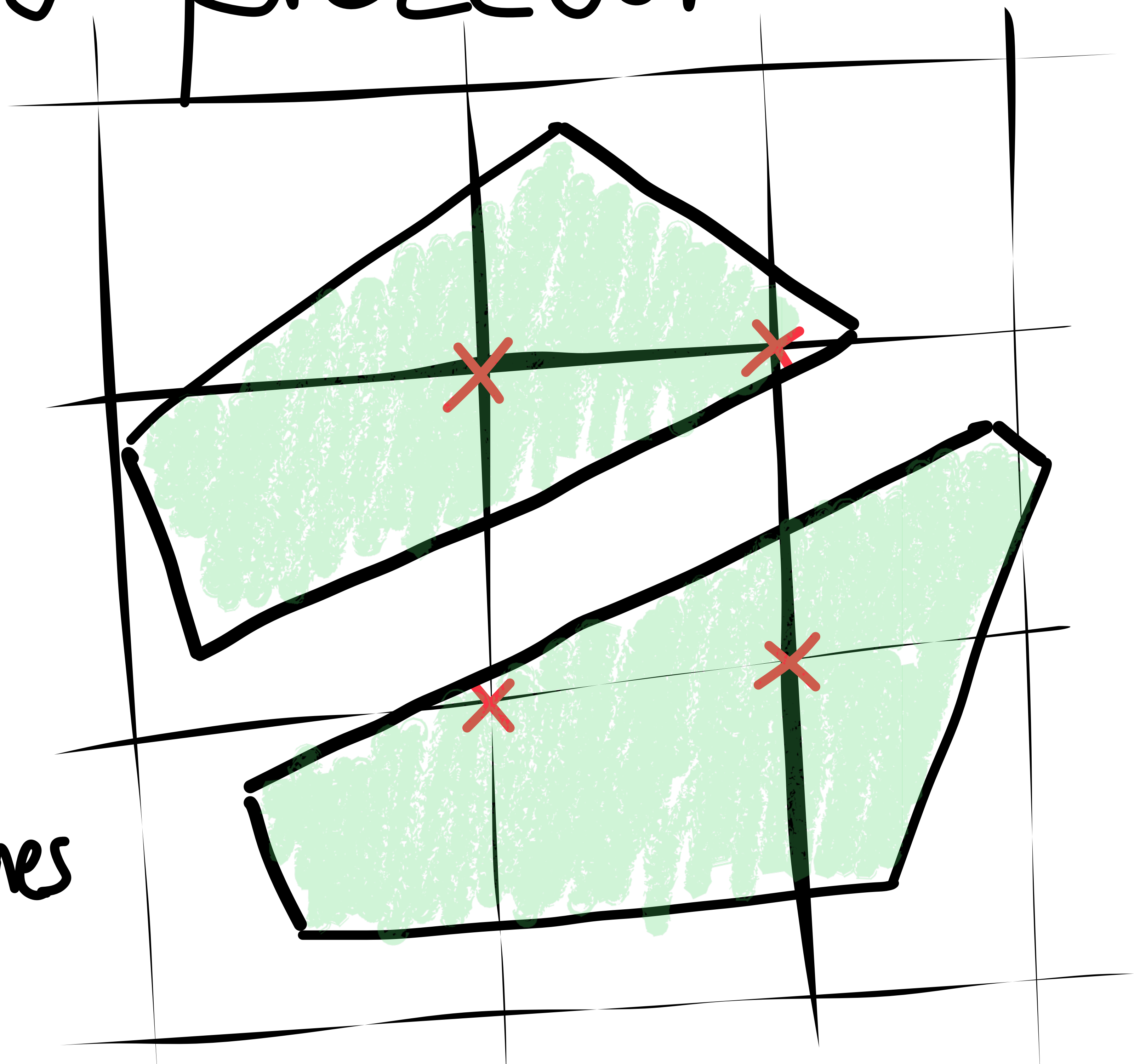

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Cut: Refine $P_1, \dots, P_k$ with additional cutting planes

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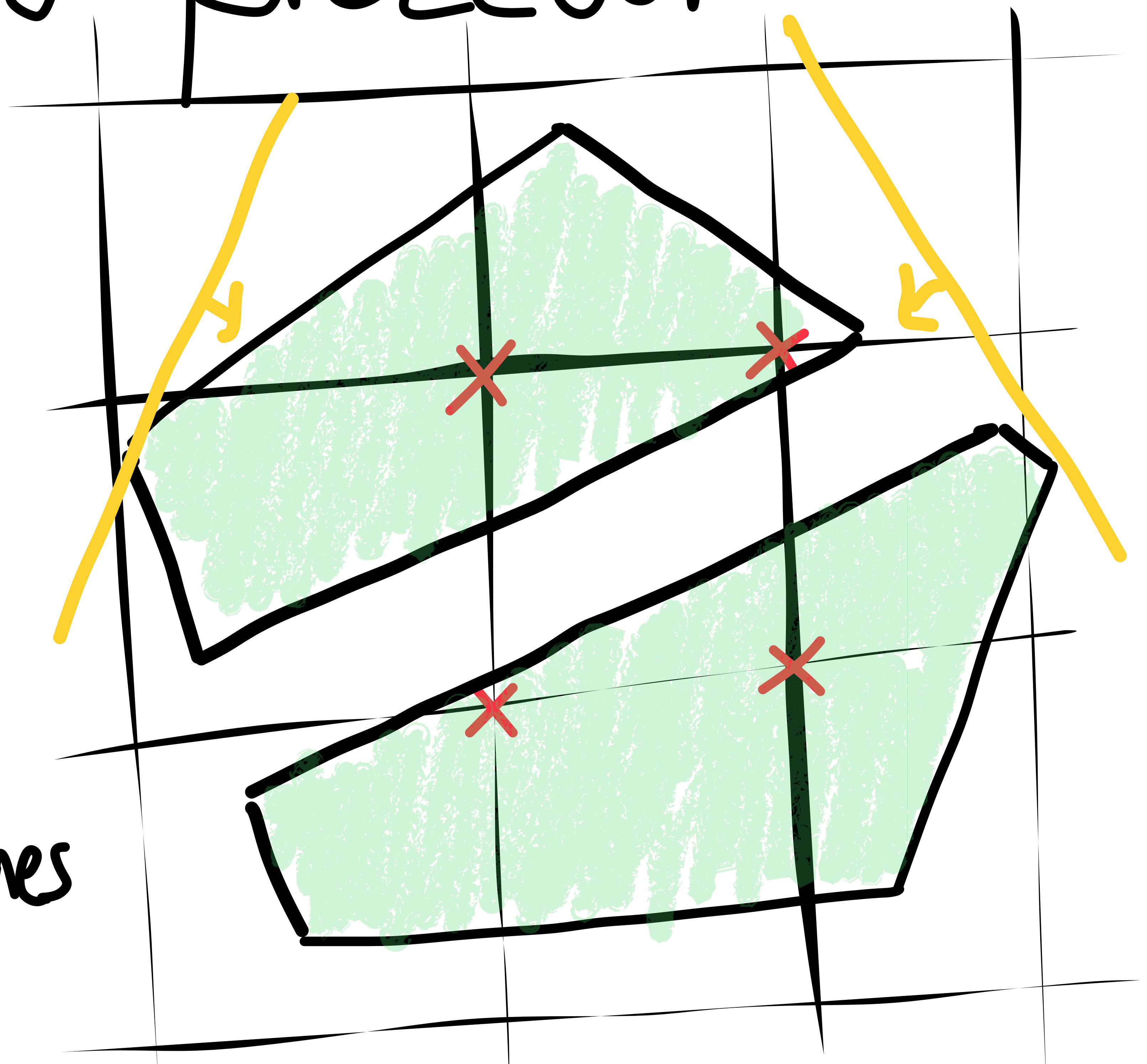

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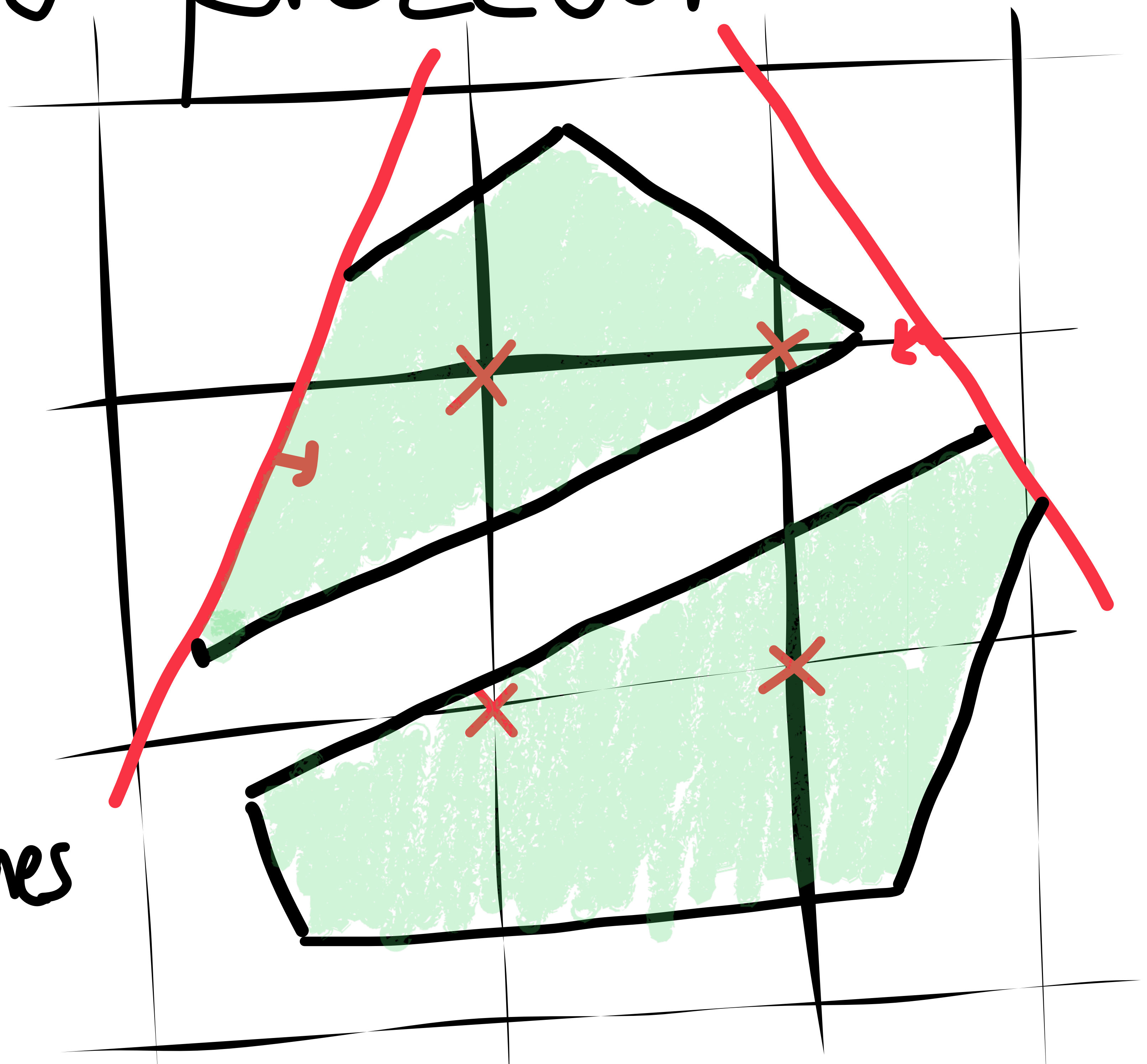

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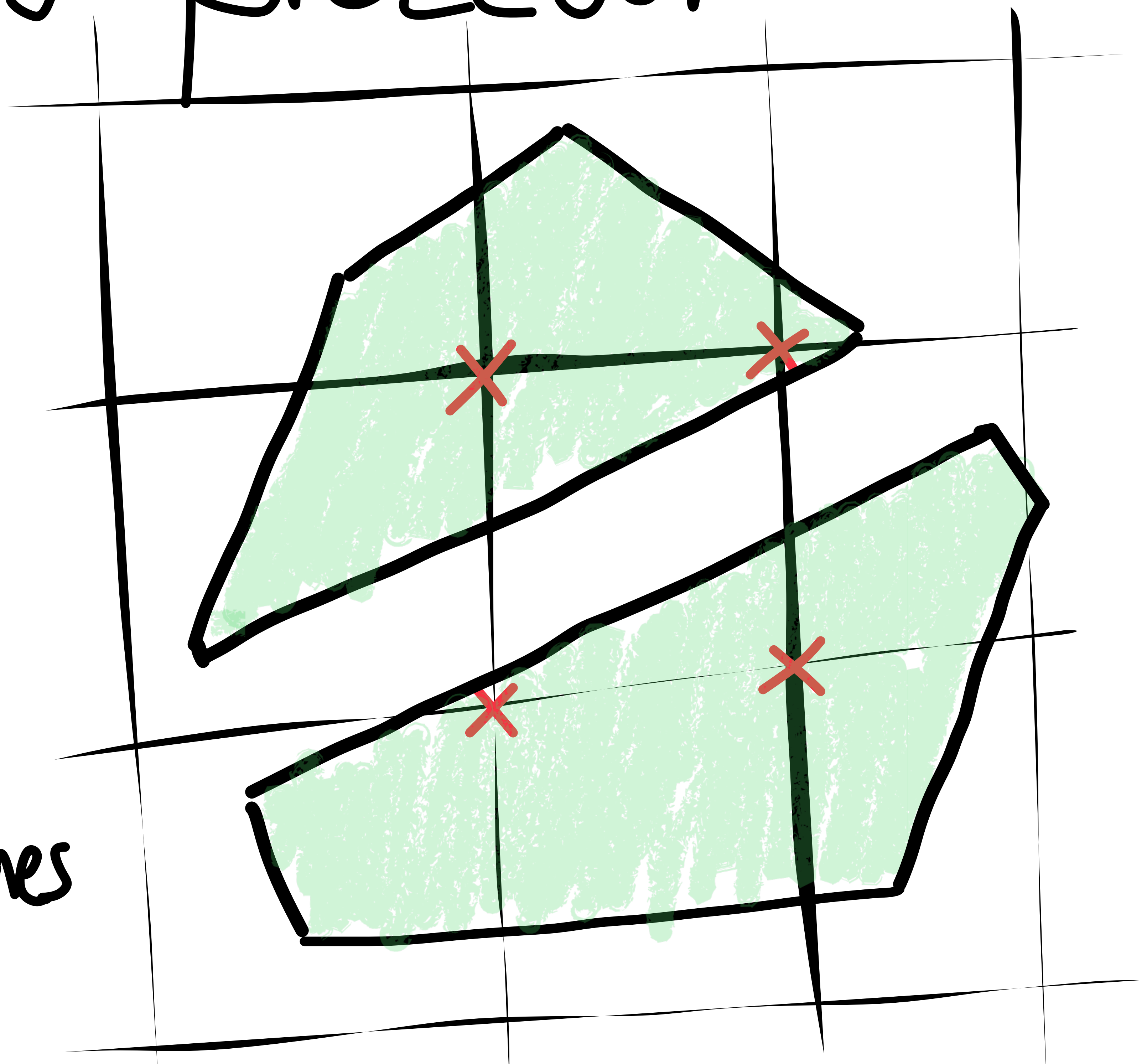

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Repeat

Stabbing Planes [BFI+18]

- Formalizes practical branch-and-cut as a proof system

Stabbing Planes [BFI+18]

- Formalizes practical branch-and-cut as a proof system
- Extends DPLL to reason about linear inequalities

DPLL Refutation

$$\{x_1 \vee x_2,\ \bar{x}_1 \vee x_2,\ x_1 \vee \bar{x}_2,\ \bar{x}_1 \vee \bar{x}_2\}$$

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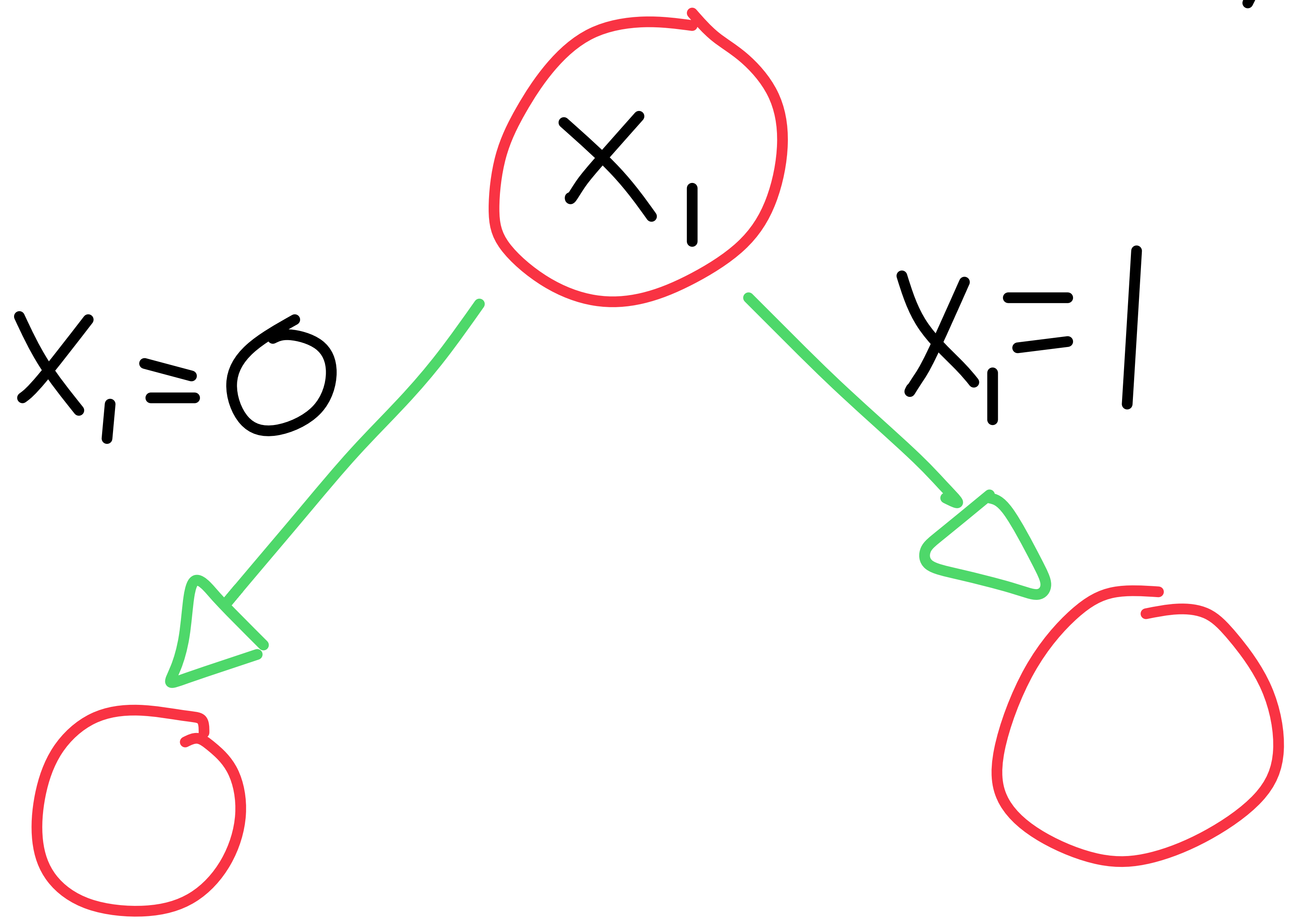

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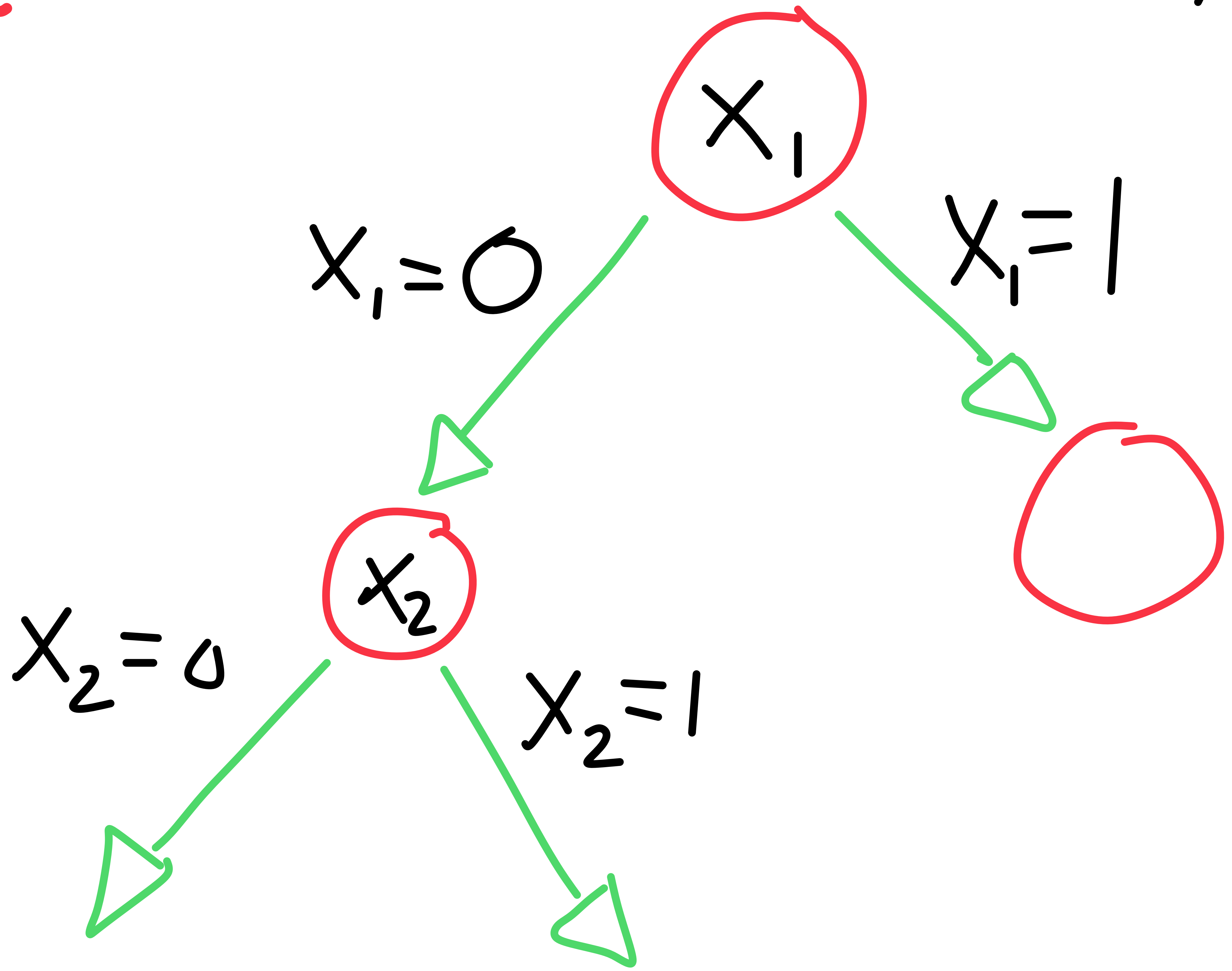

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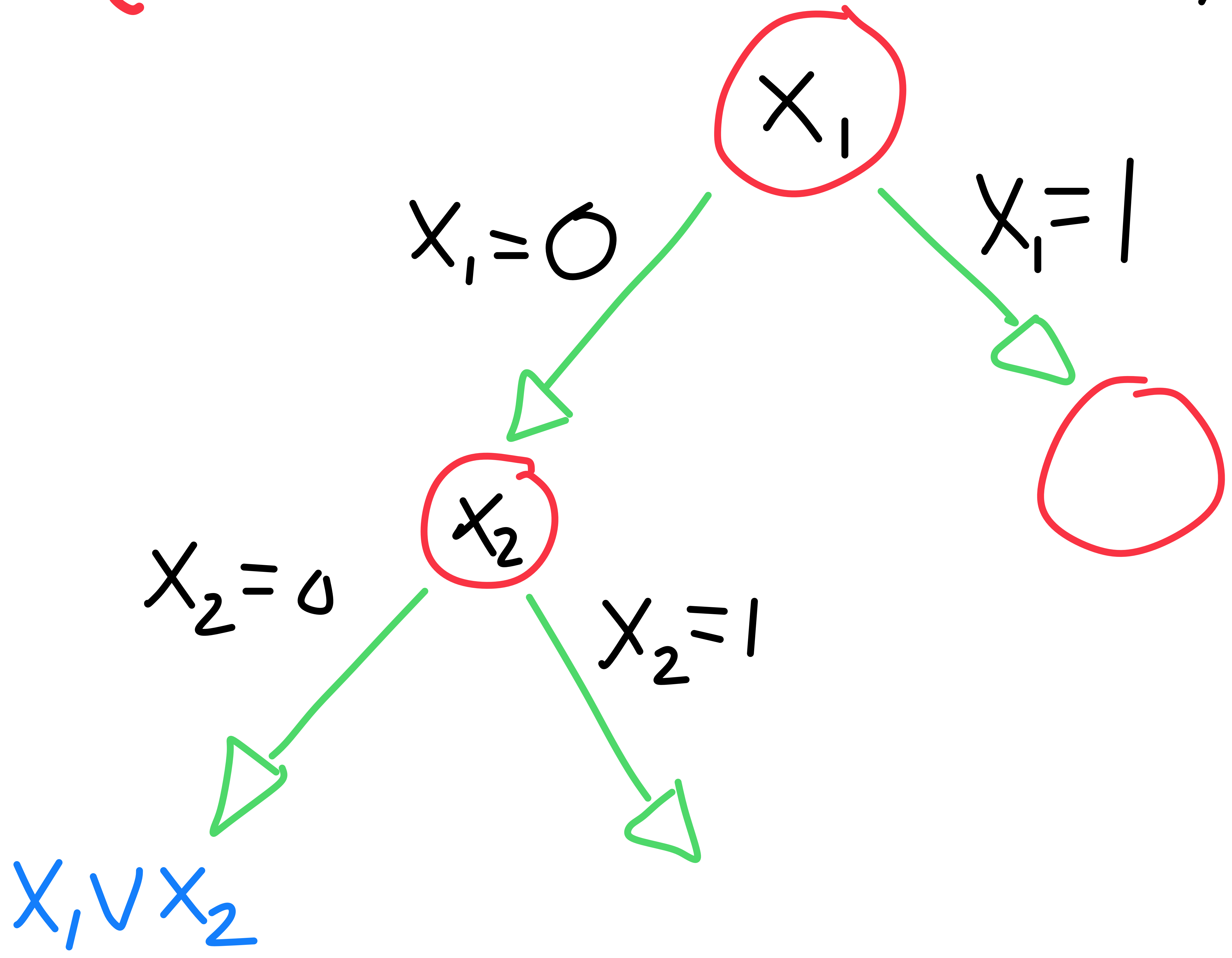

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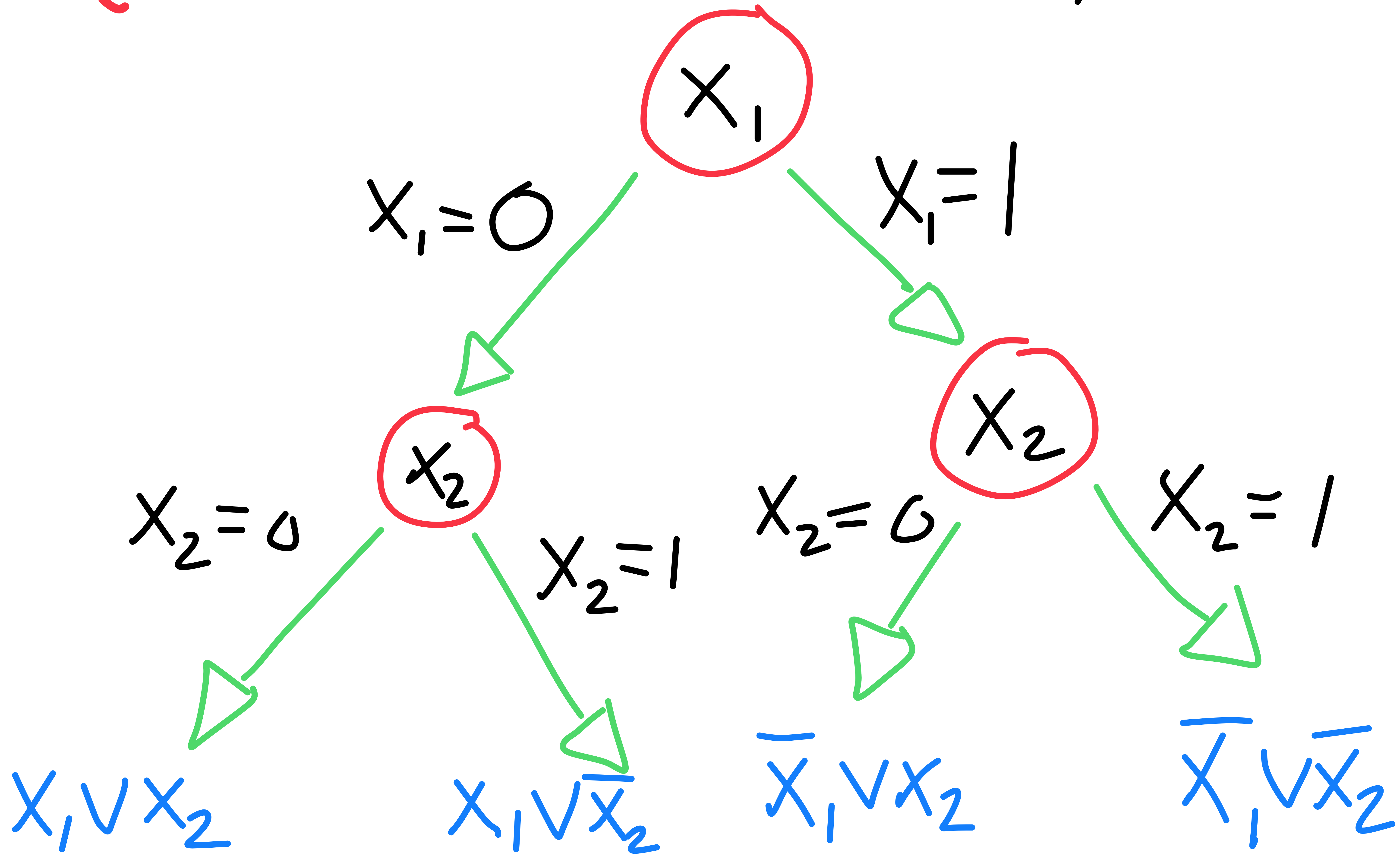

DPLL as Polytopes

$$P = \{ x_1 + x_2 \geq 1,\ x_1 - x_2 \geq 0,\ x_2 - x_1 \geq 0,\ -x_1 - x_2 \geq -1,\ 0 \leq x_i \leq 1 \}$$

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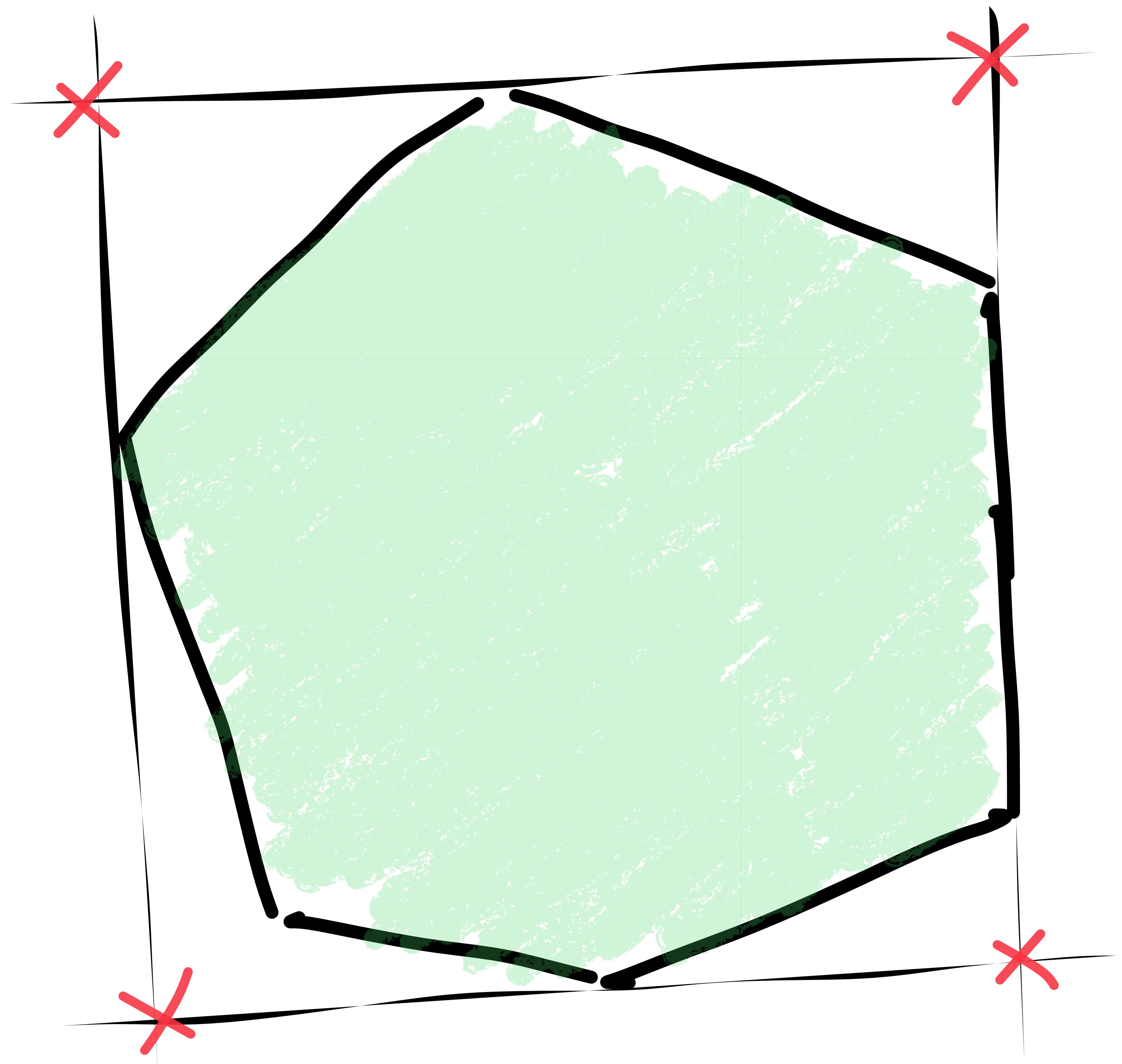
P



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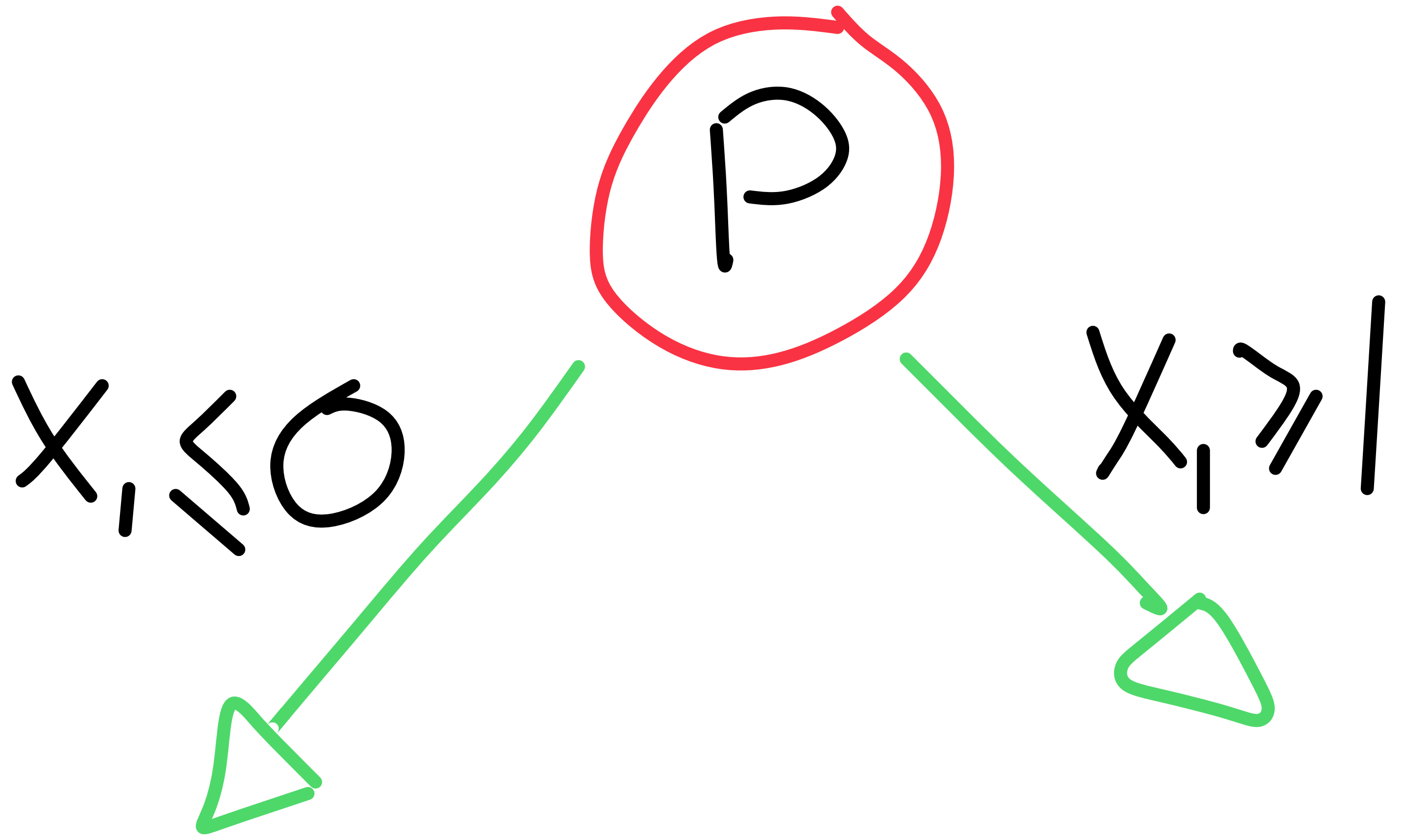

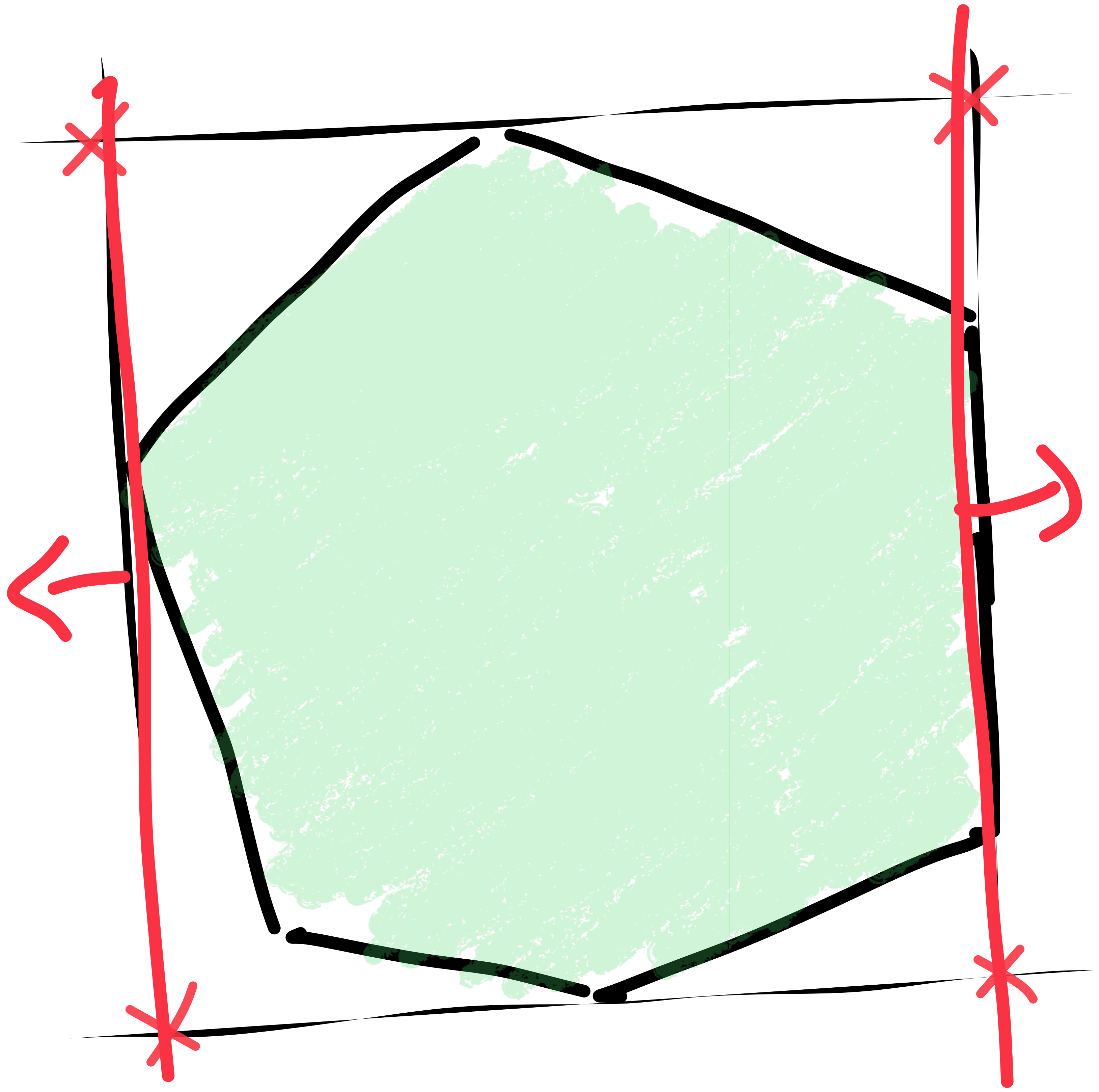

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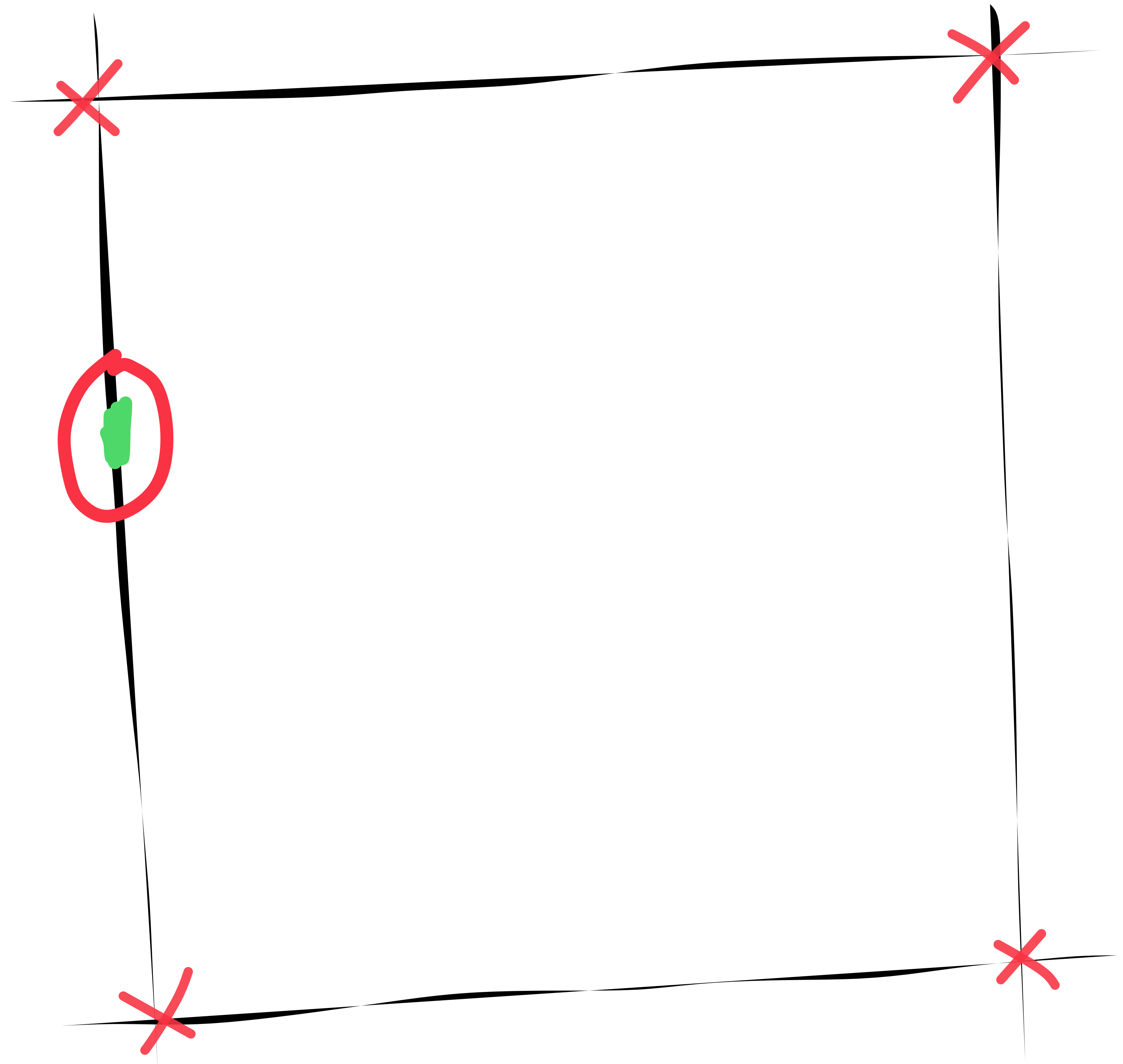
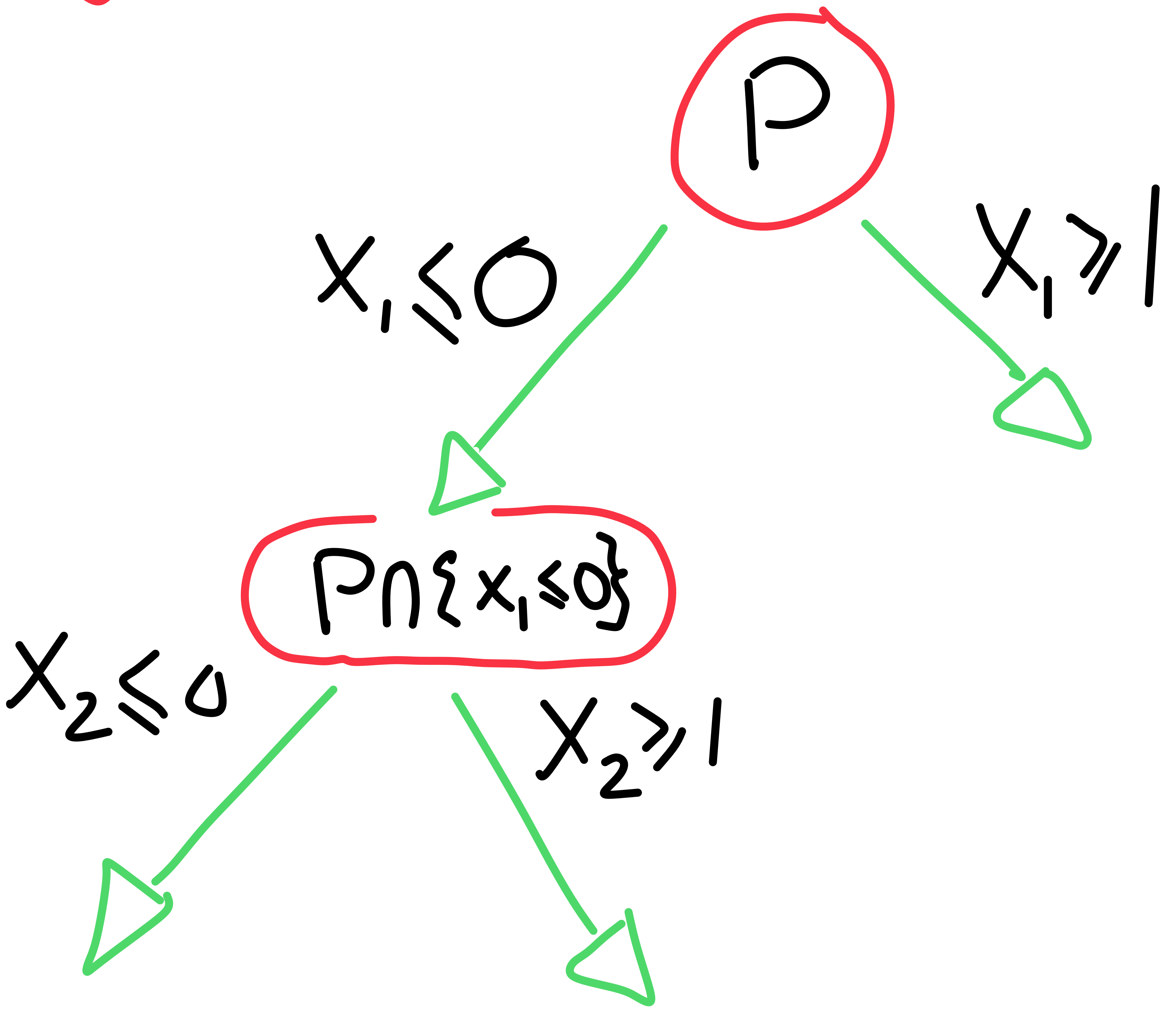
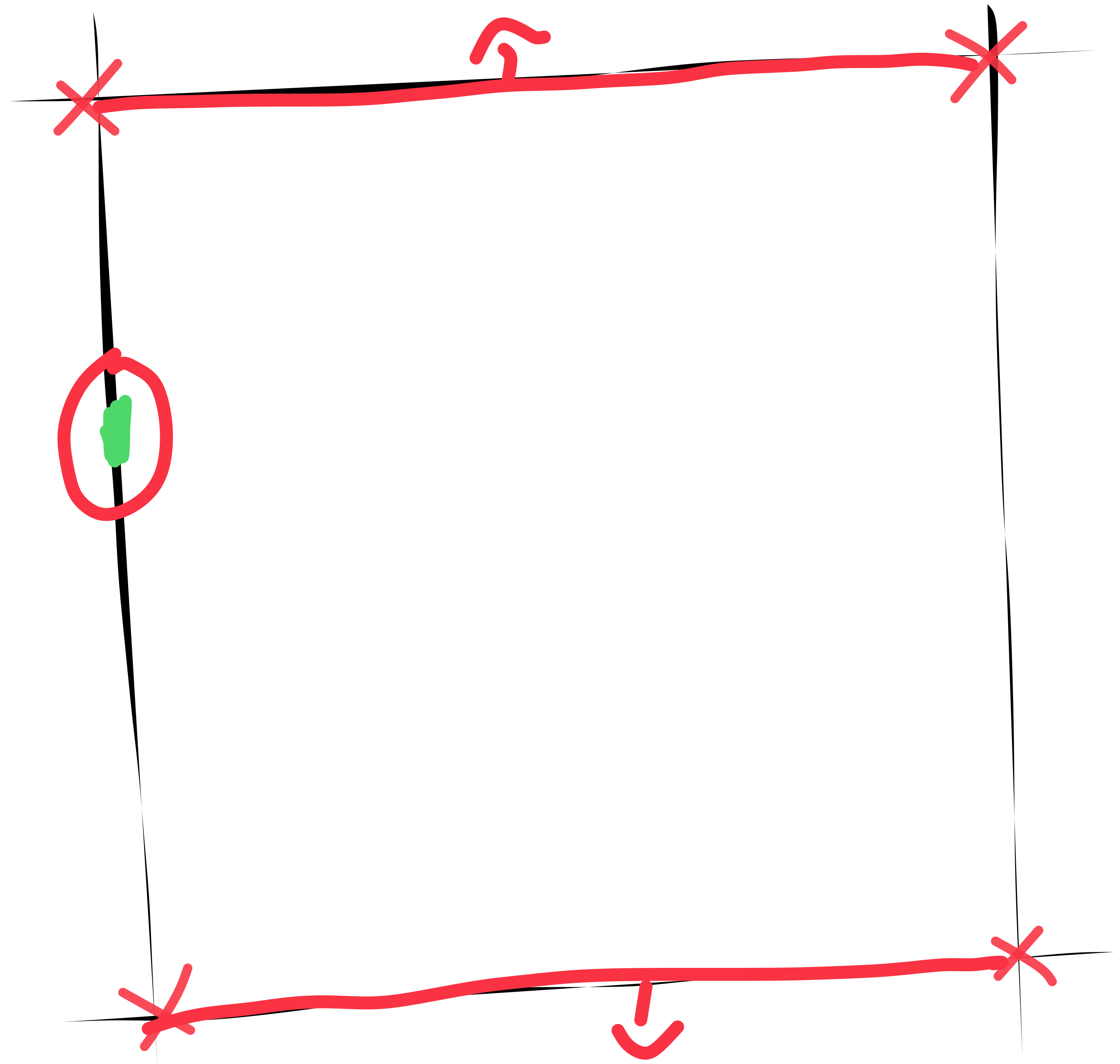
$$\begin{tikzcd}
 & P \arrow[dl, "x_1 \leq 0"'] \arrow[dr, "x_1 \geq 1"] & \\
P \cap \{x_1 \leq 0\} & & {}
\end{tikzcd}$$



DPLL as Polytopes

$$P = \{ x_1 + x_2 \geq 1,\ x_1 - x_2 \geq 0,\ x_2 - x_1 \geq 0,\ -x_1 - x_2 \geq -1,\ 0 \leq x_i \leq 1 \}$$





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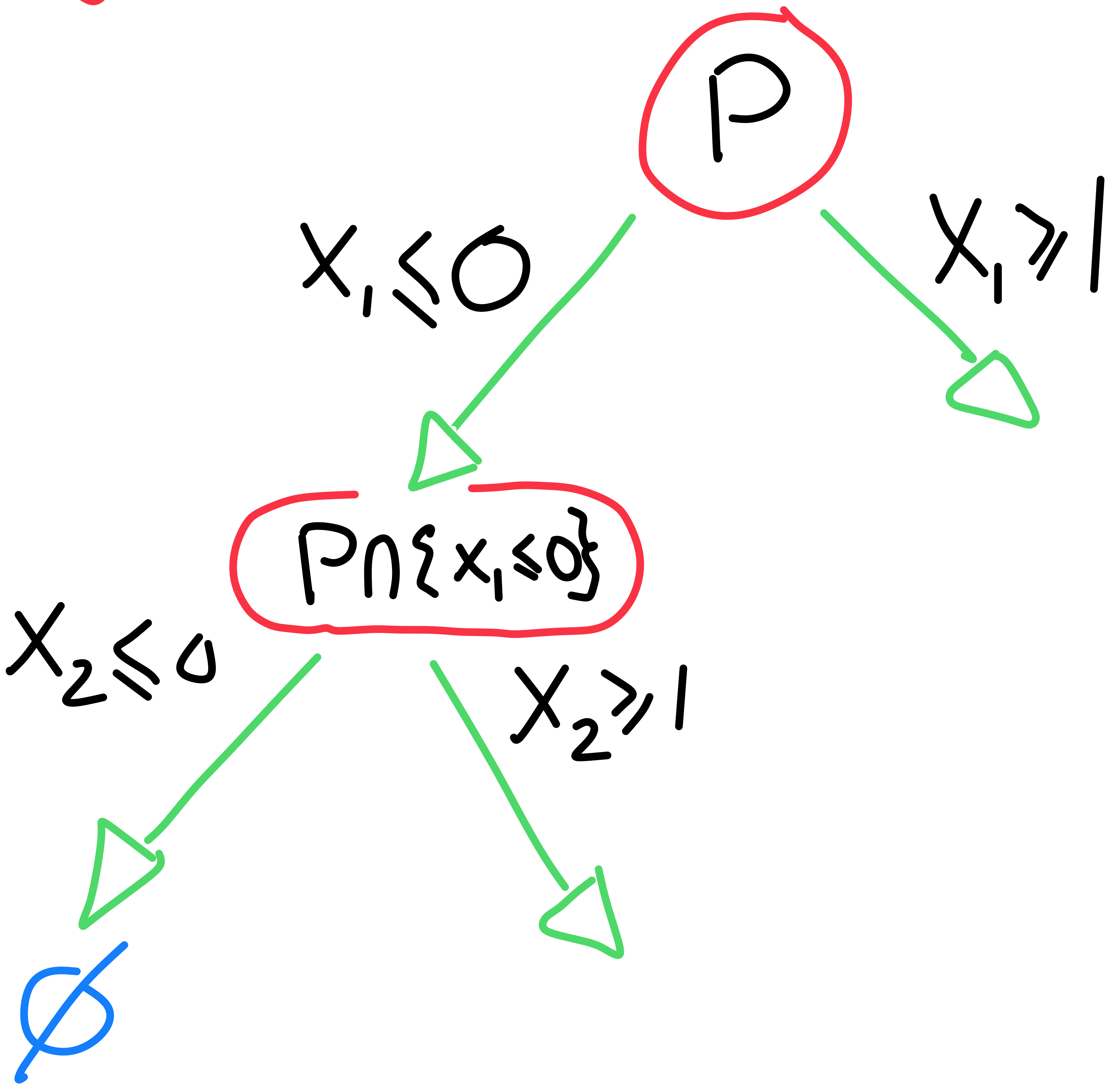

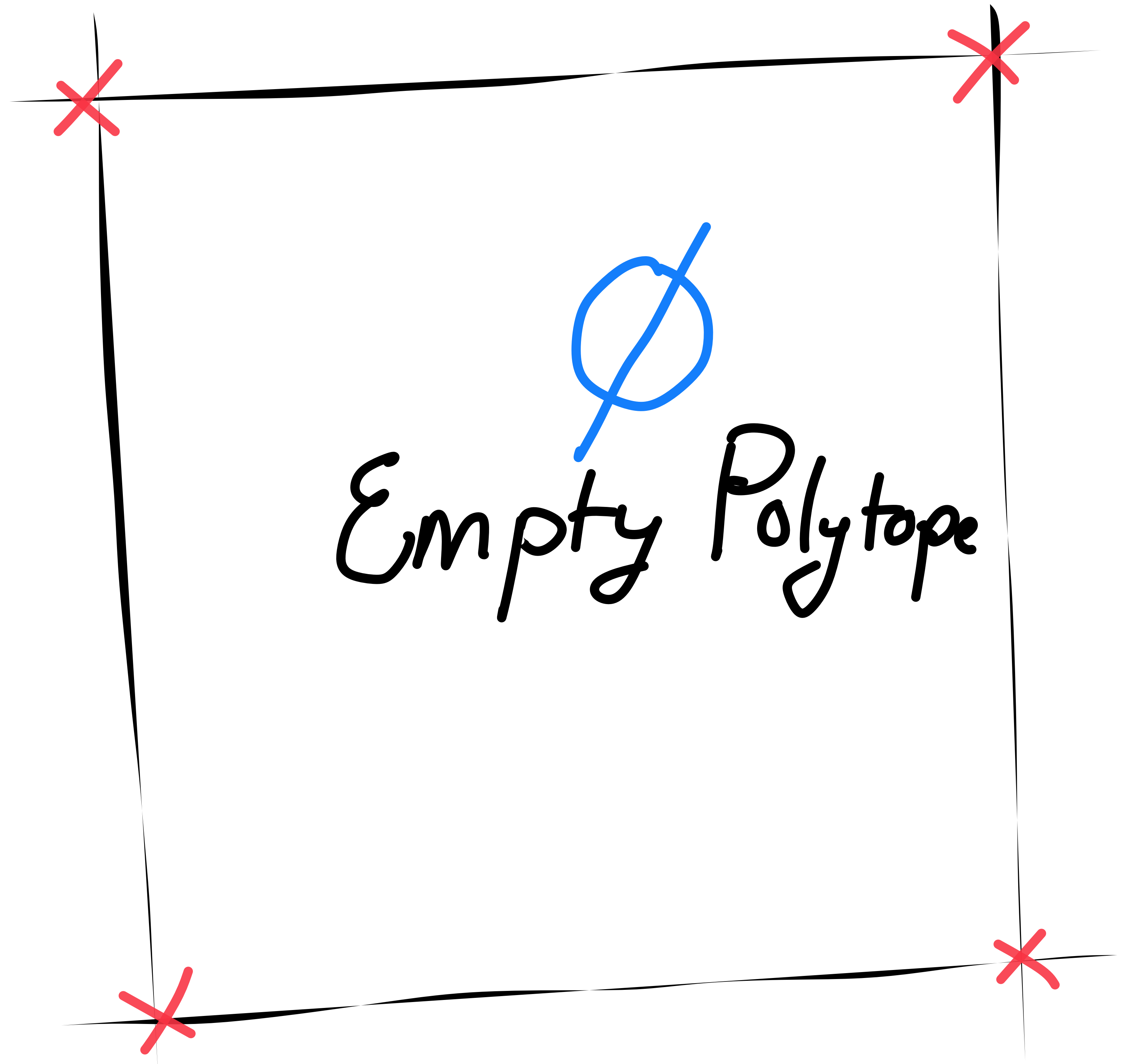

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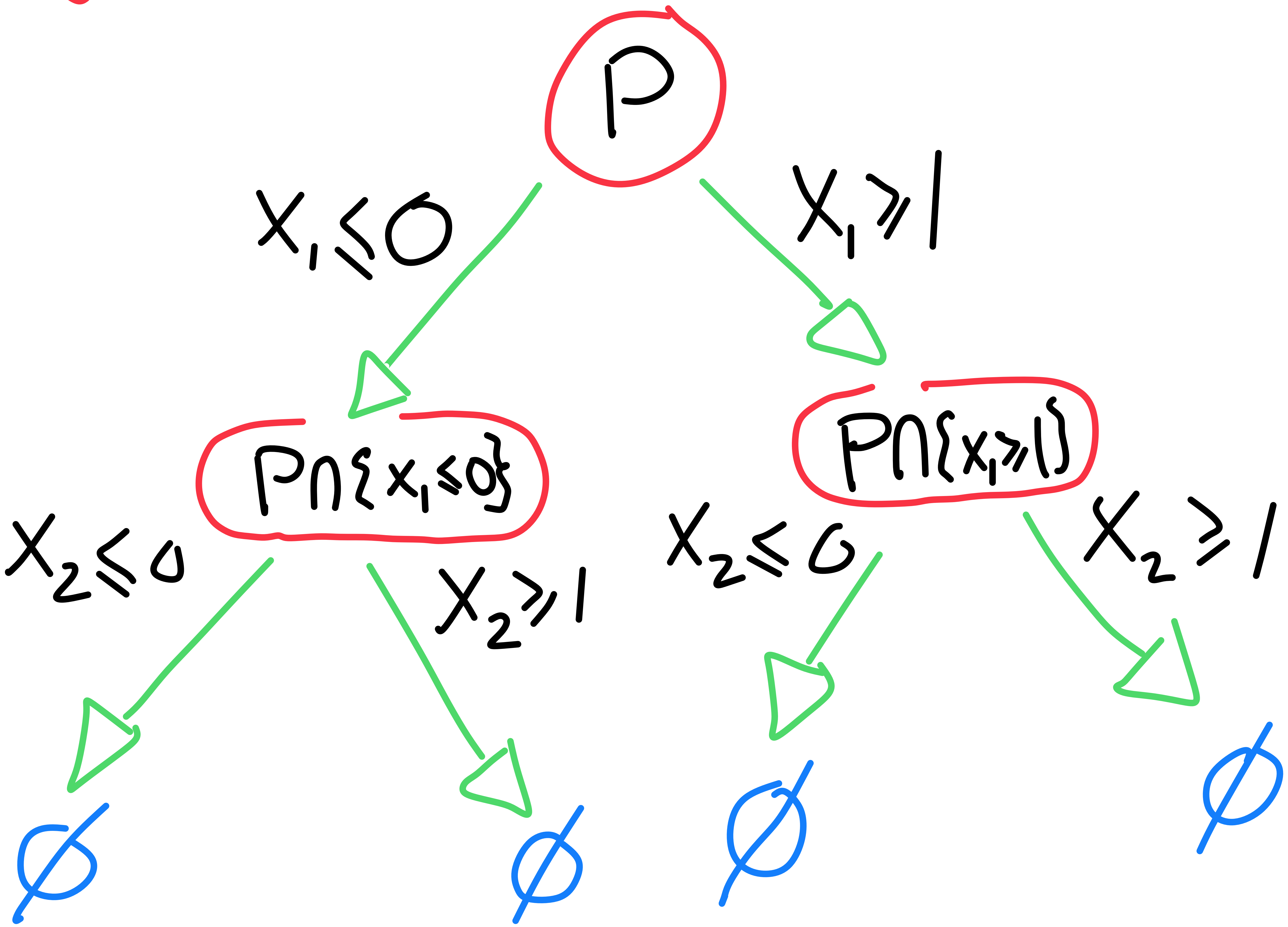

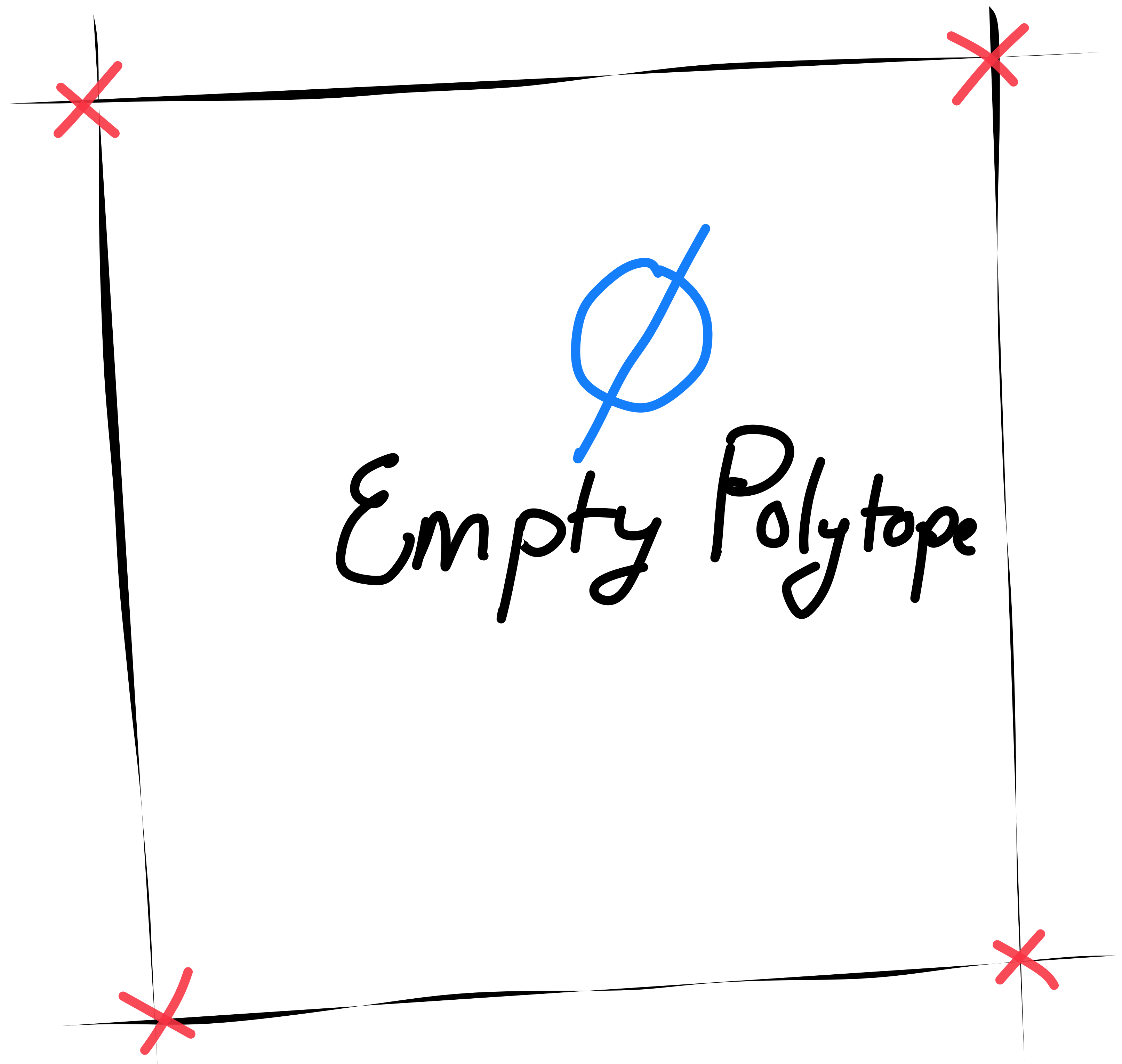

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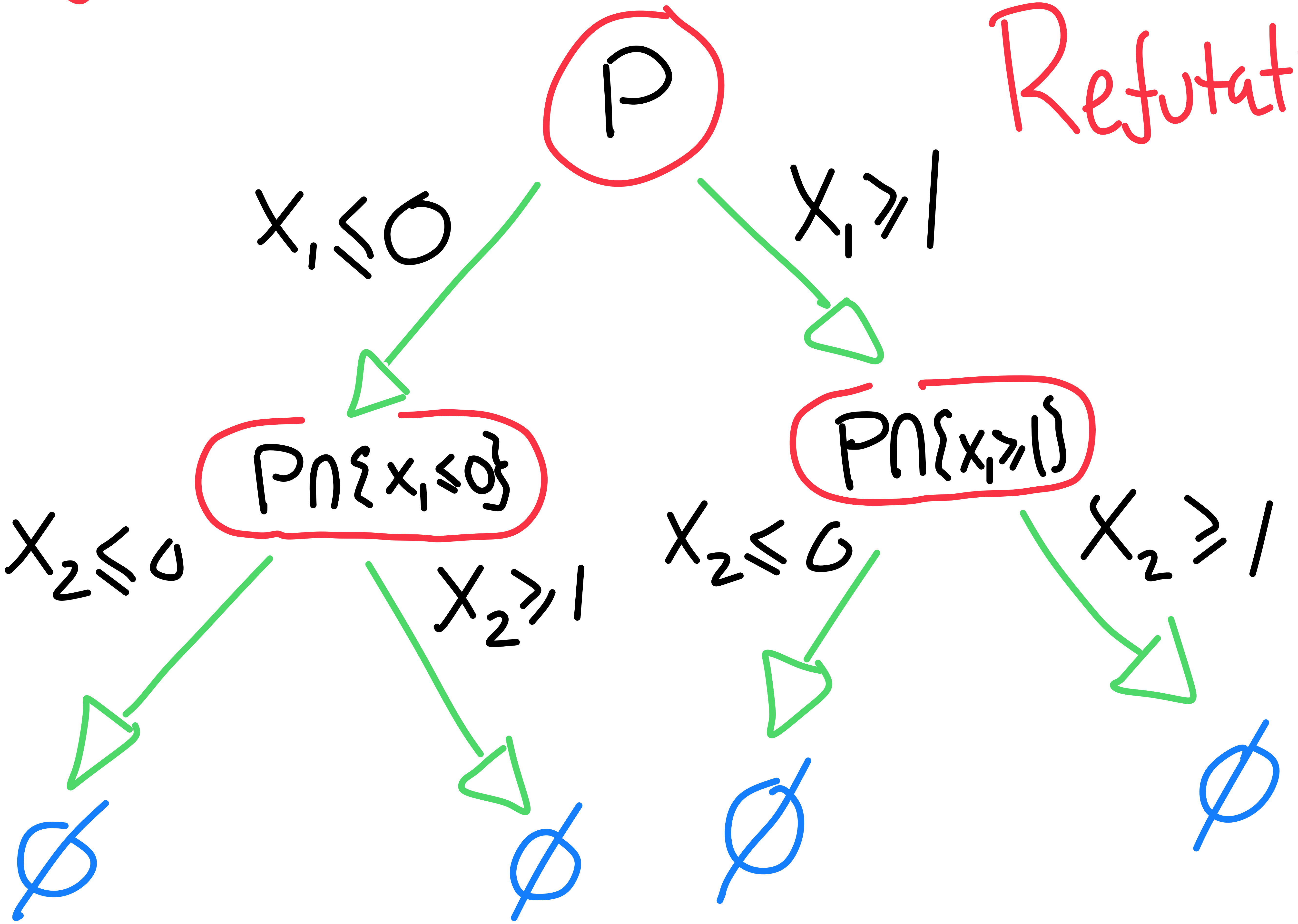
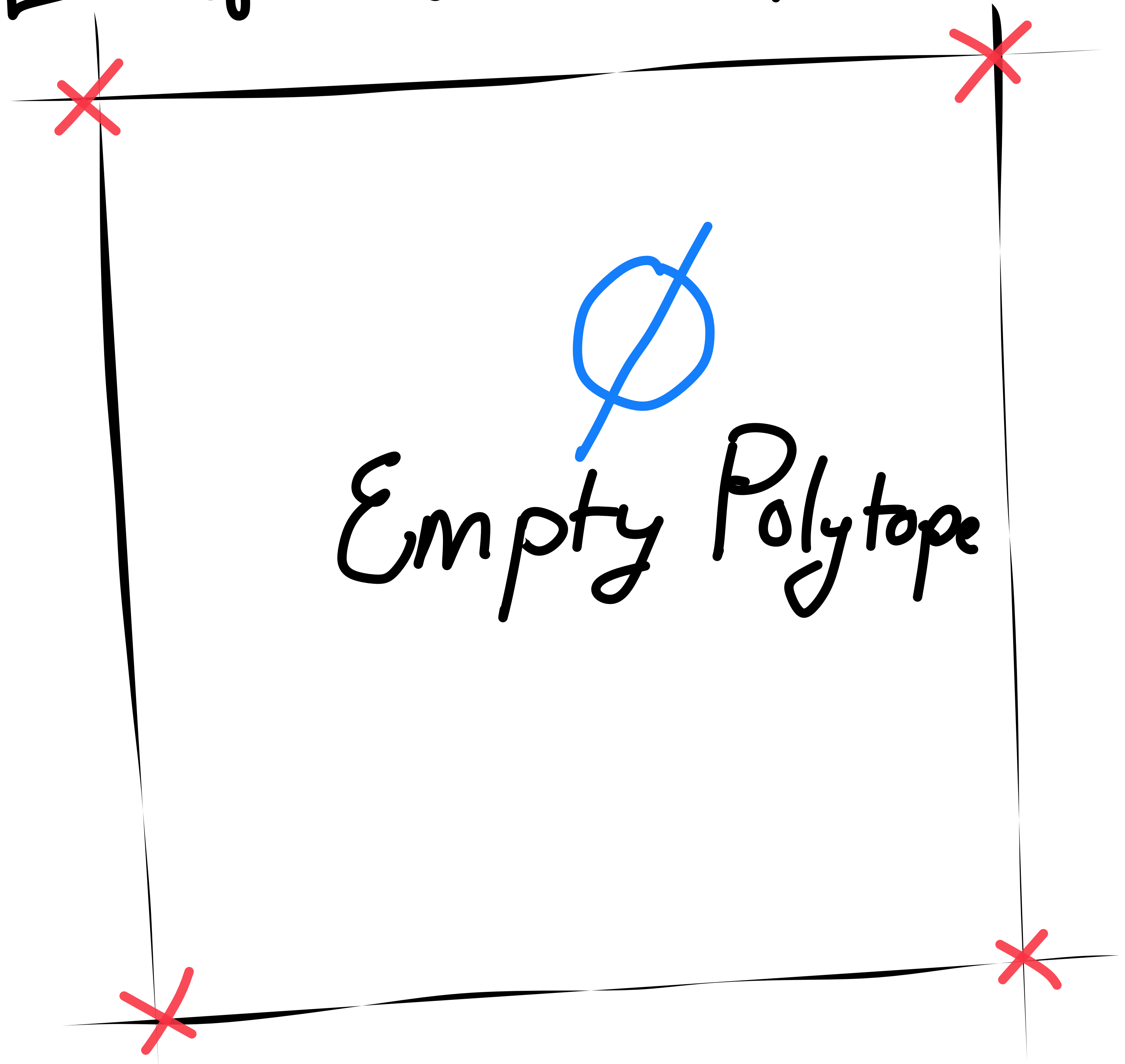


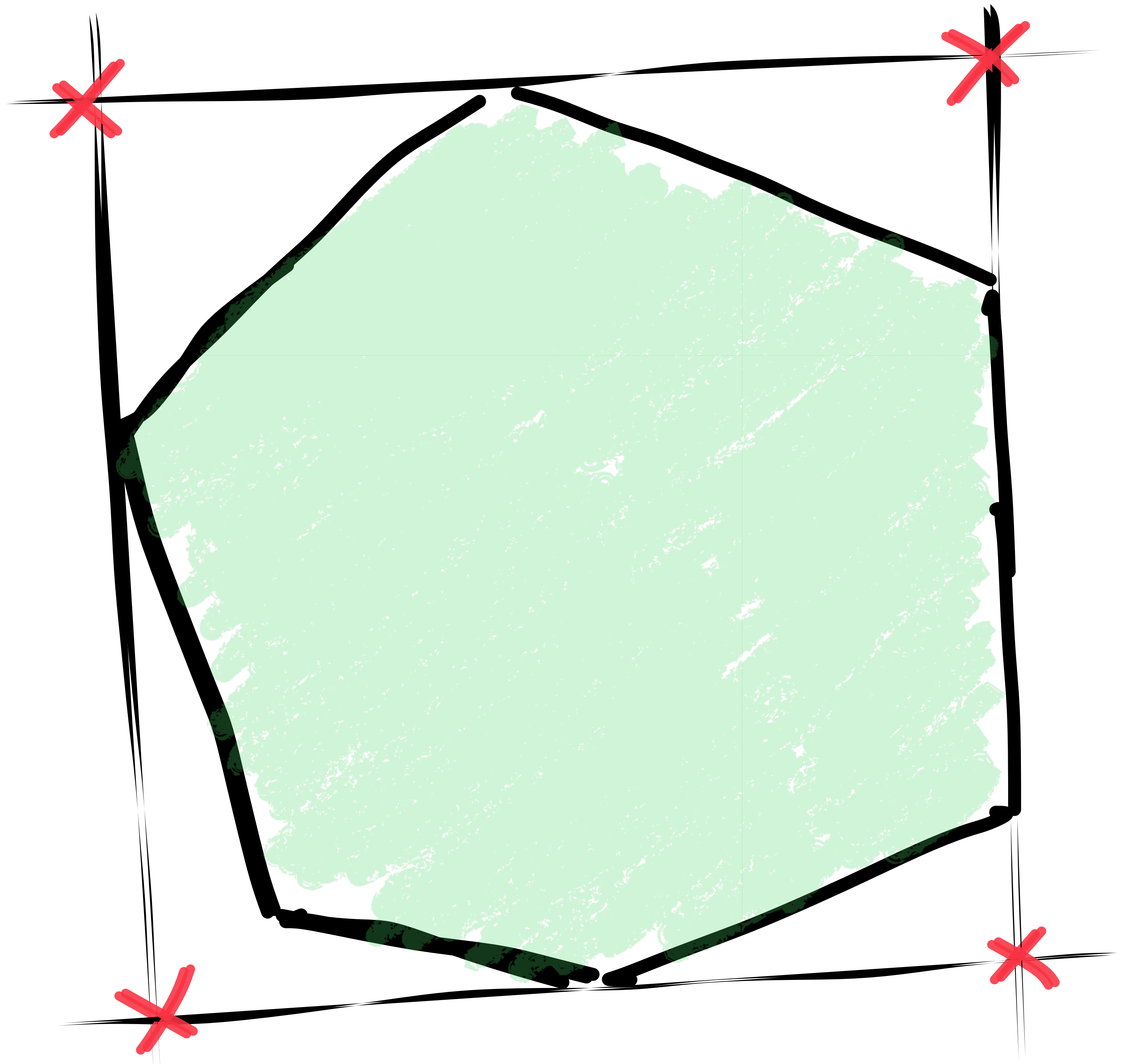
Refutation! $\emptyset$ Derived at every Leaf of the tree



Stabbing Planes

$P = \{Ax \geq b\}$ s.t. $P \cap \mathbb{Z}^n = \emptyset$.

P



Stabbing Planes

"query arbitrary linear inequalities"

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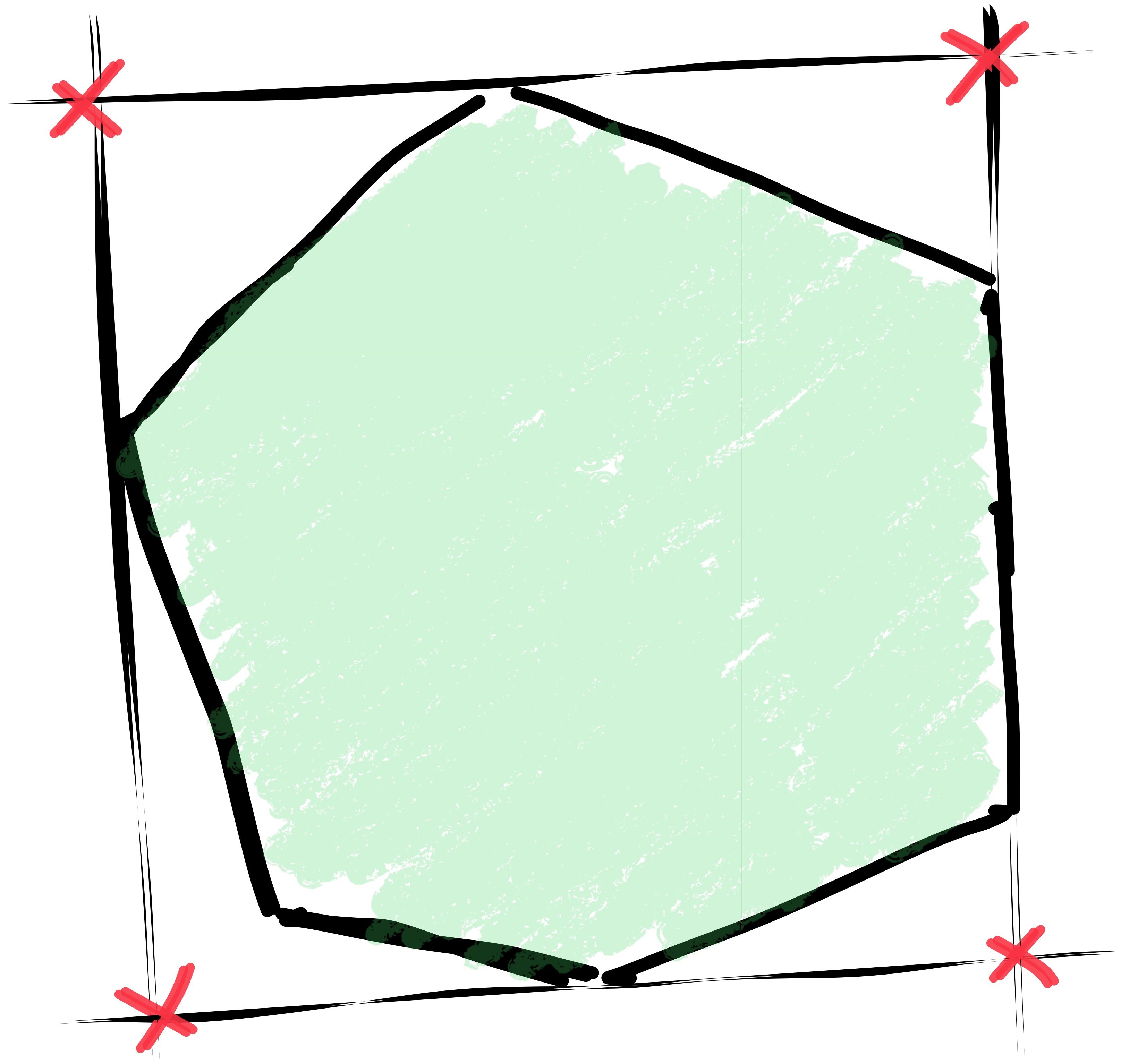

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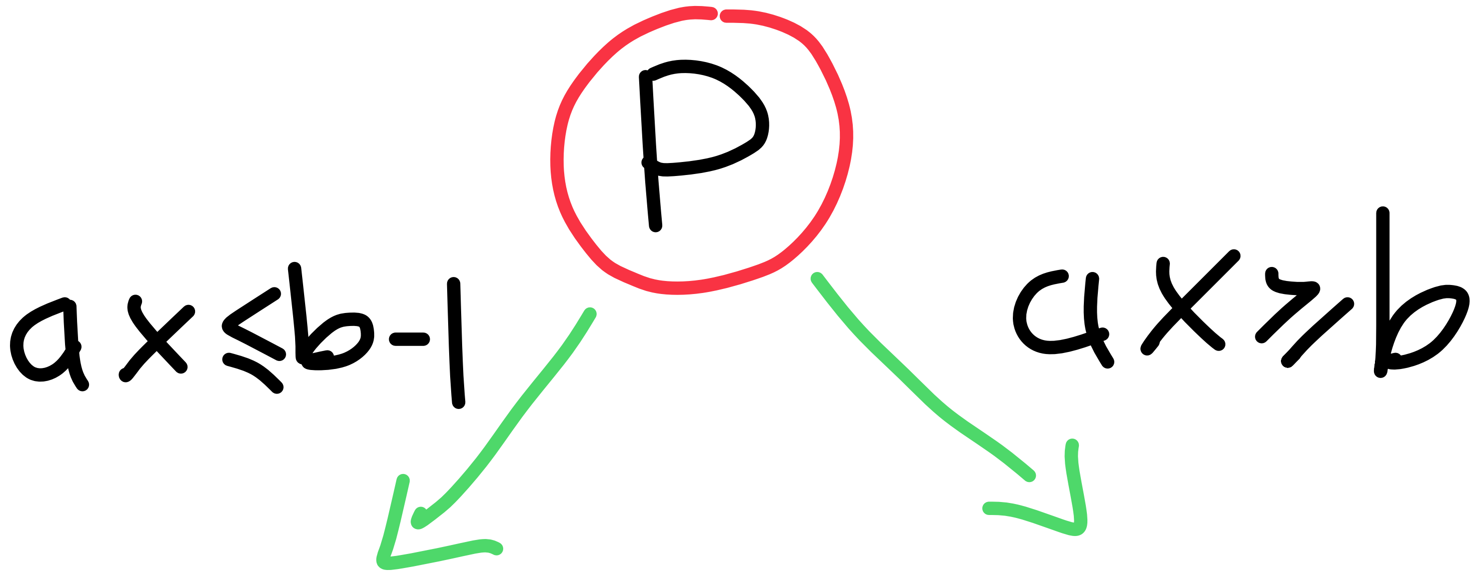

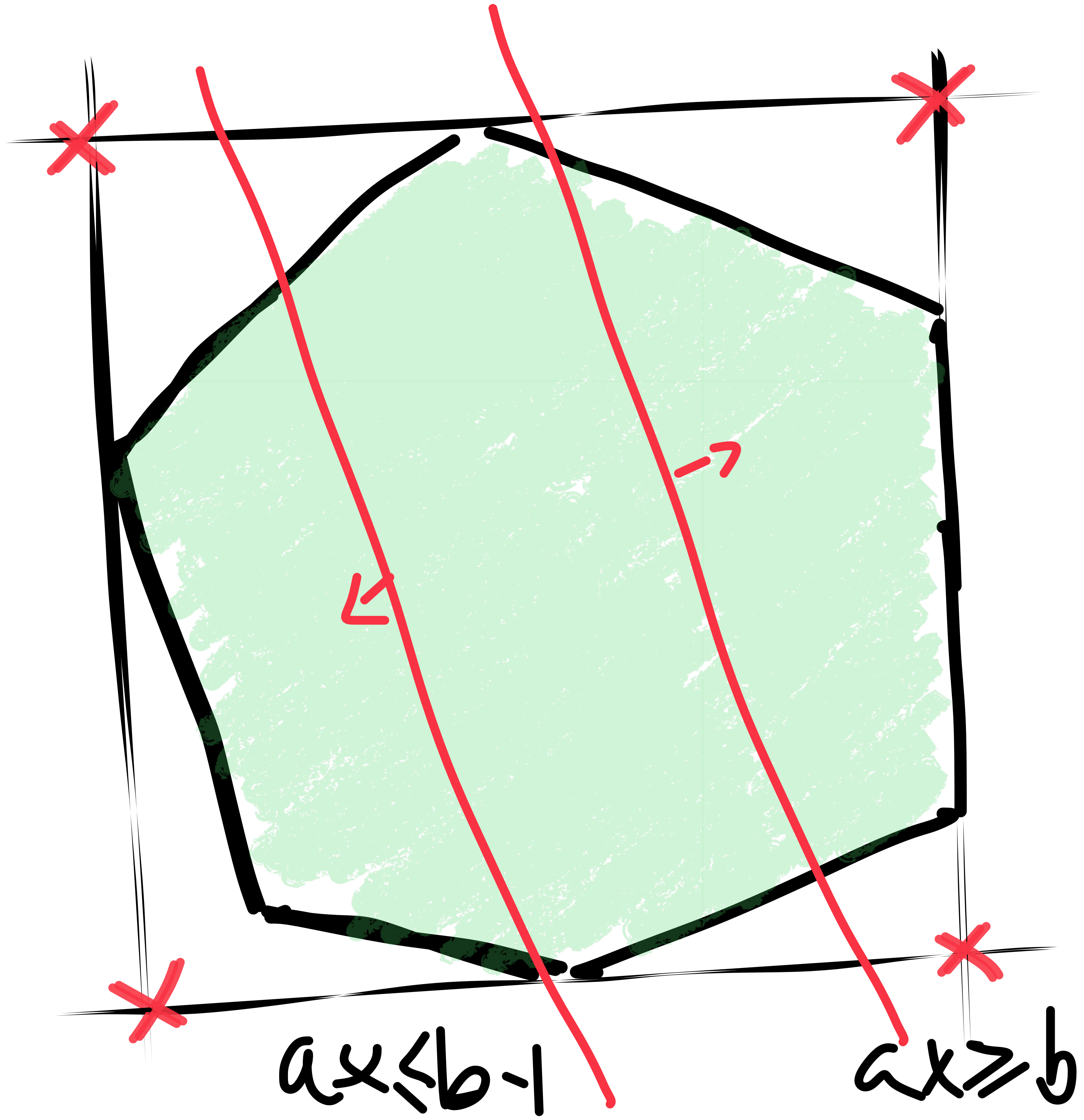

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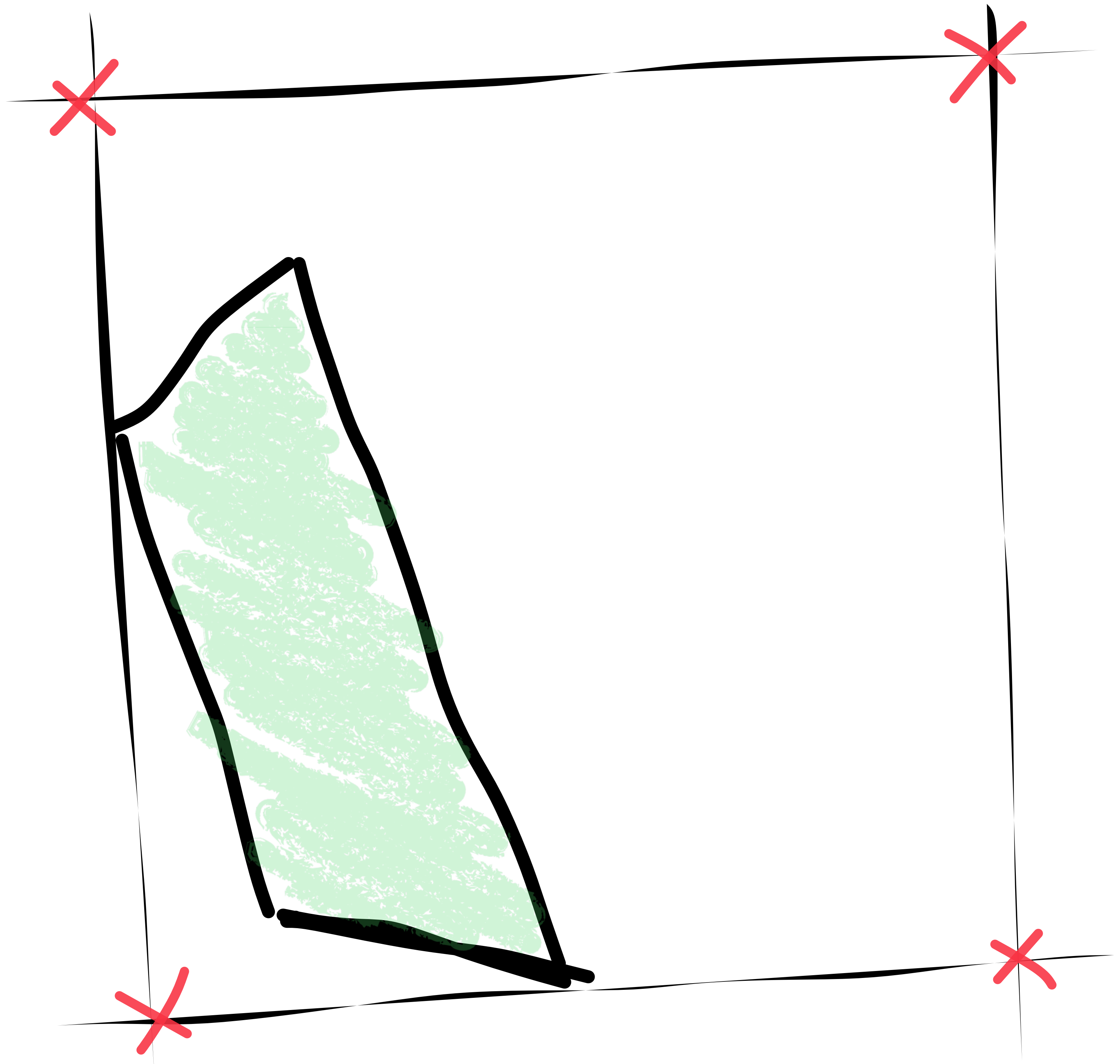

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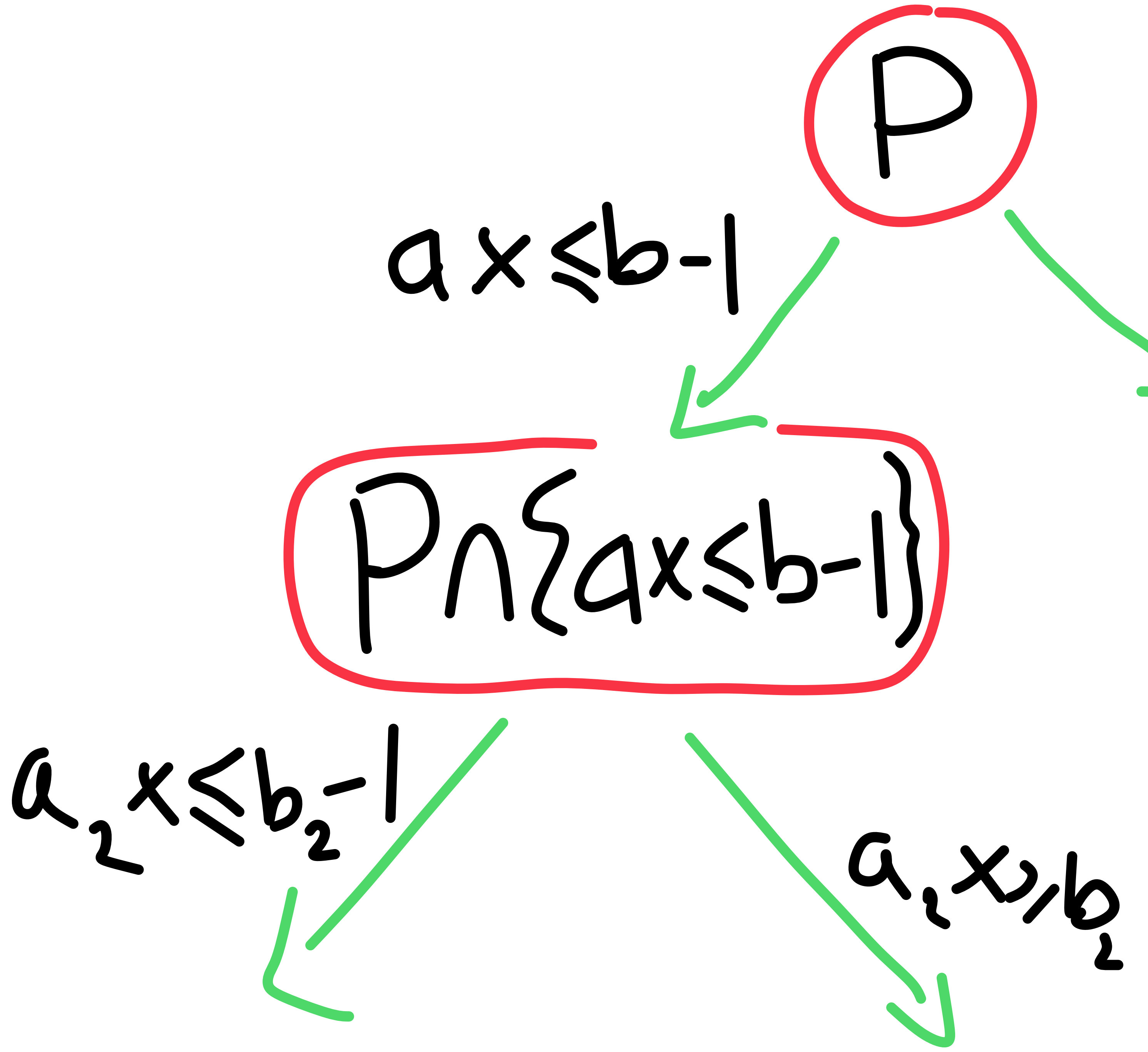

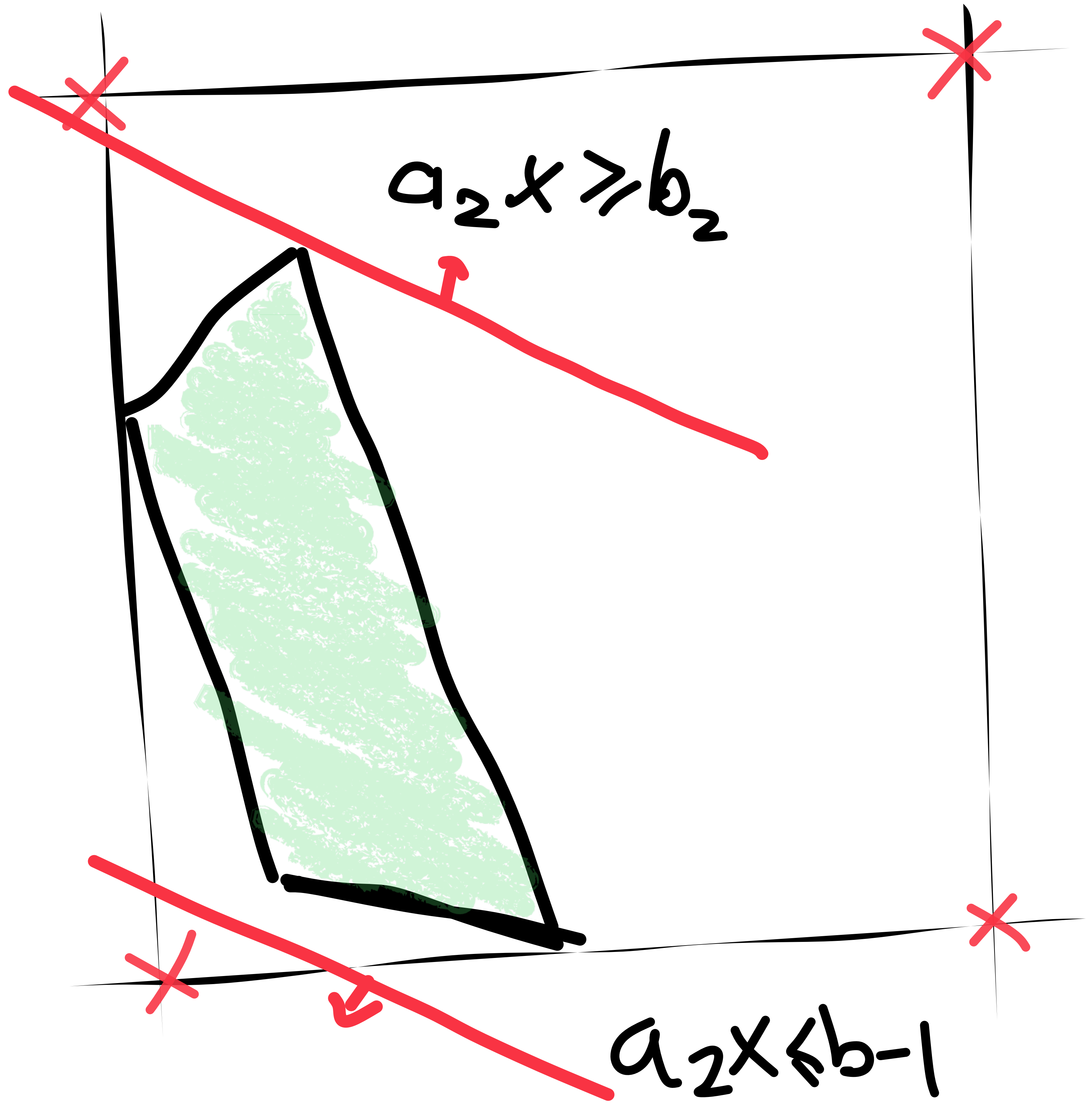

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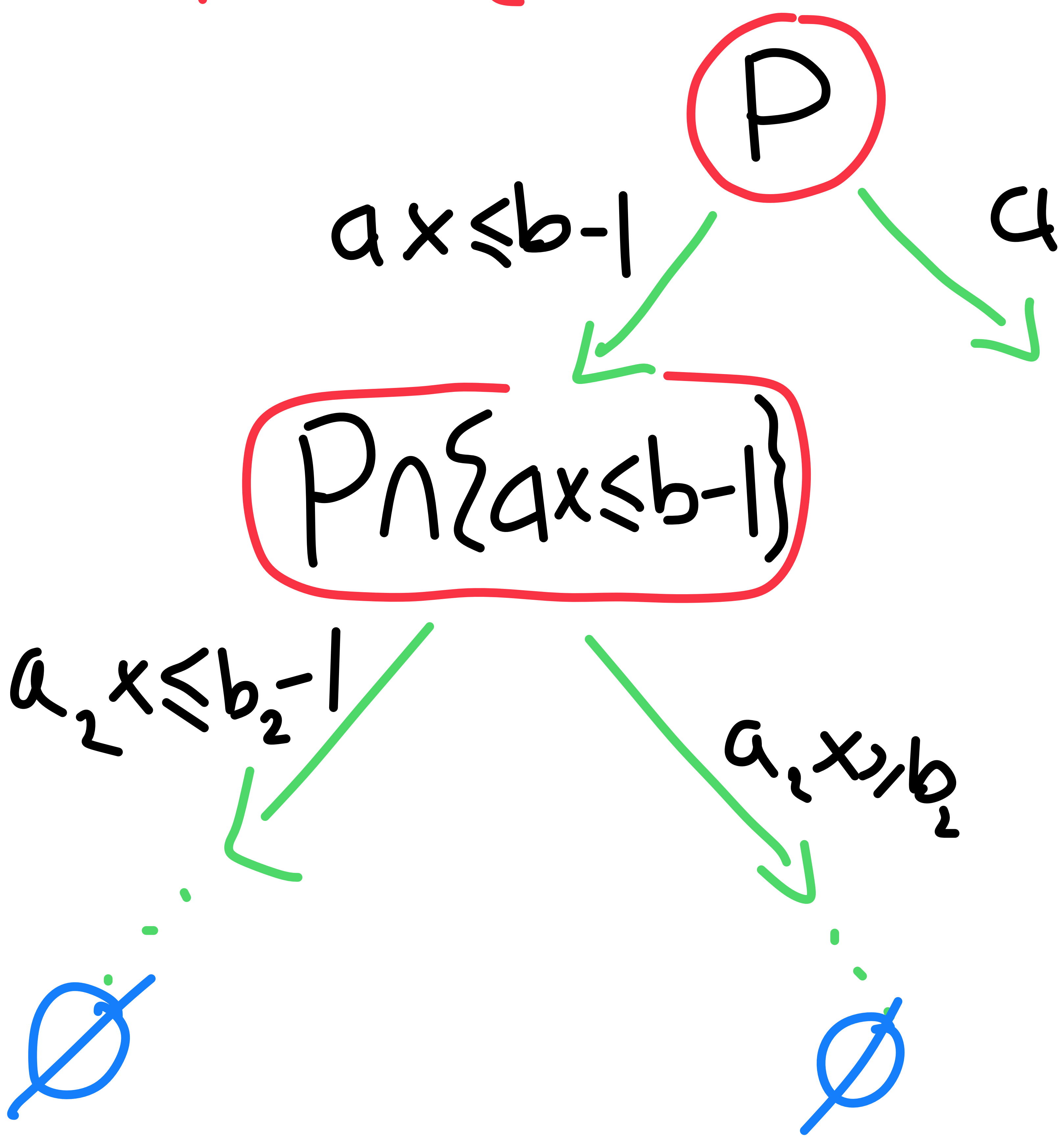

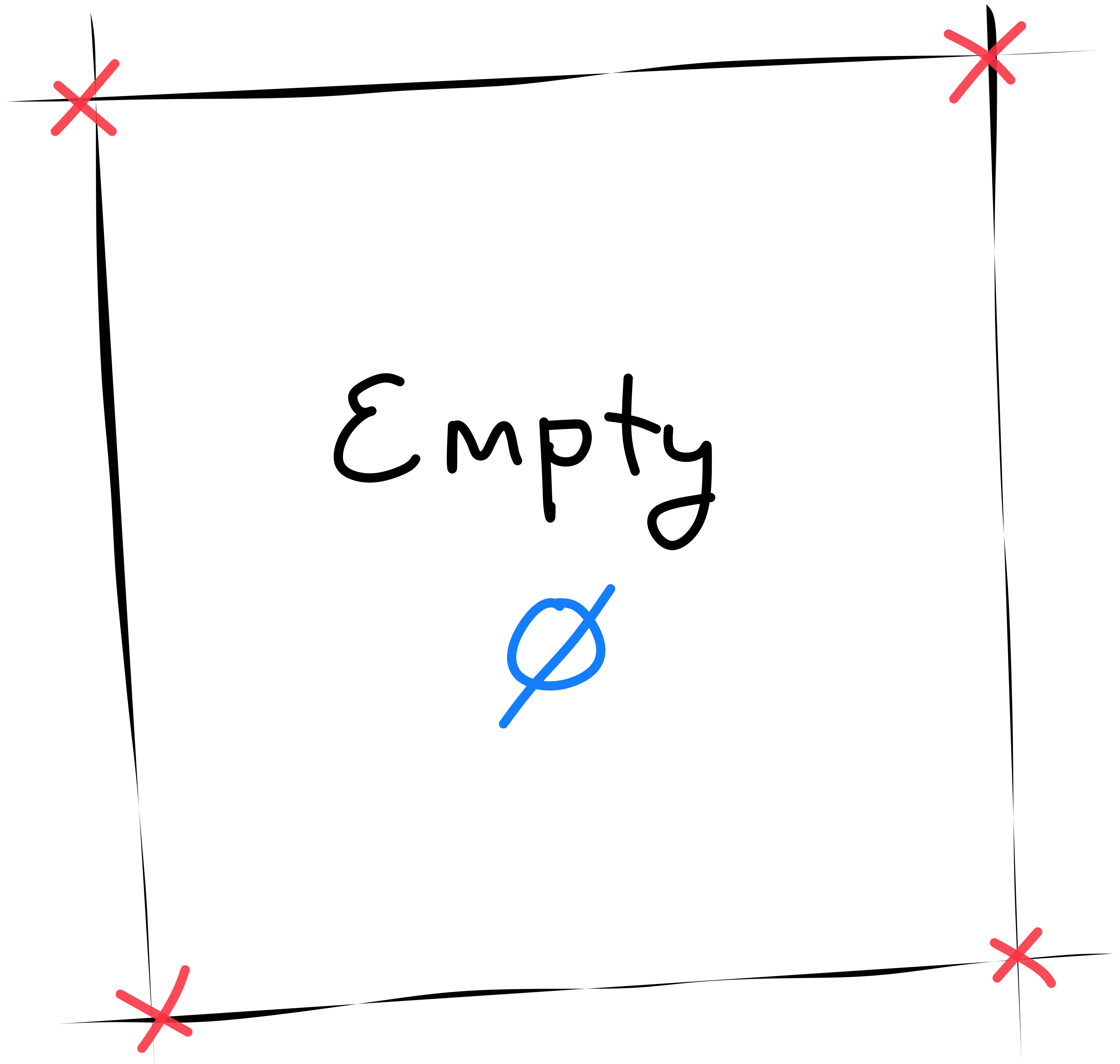

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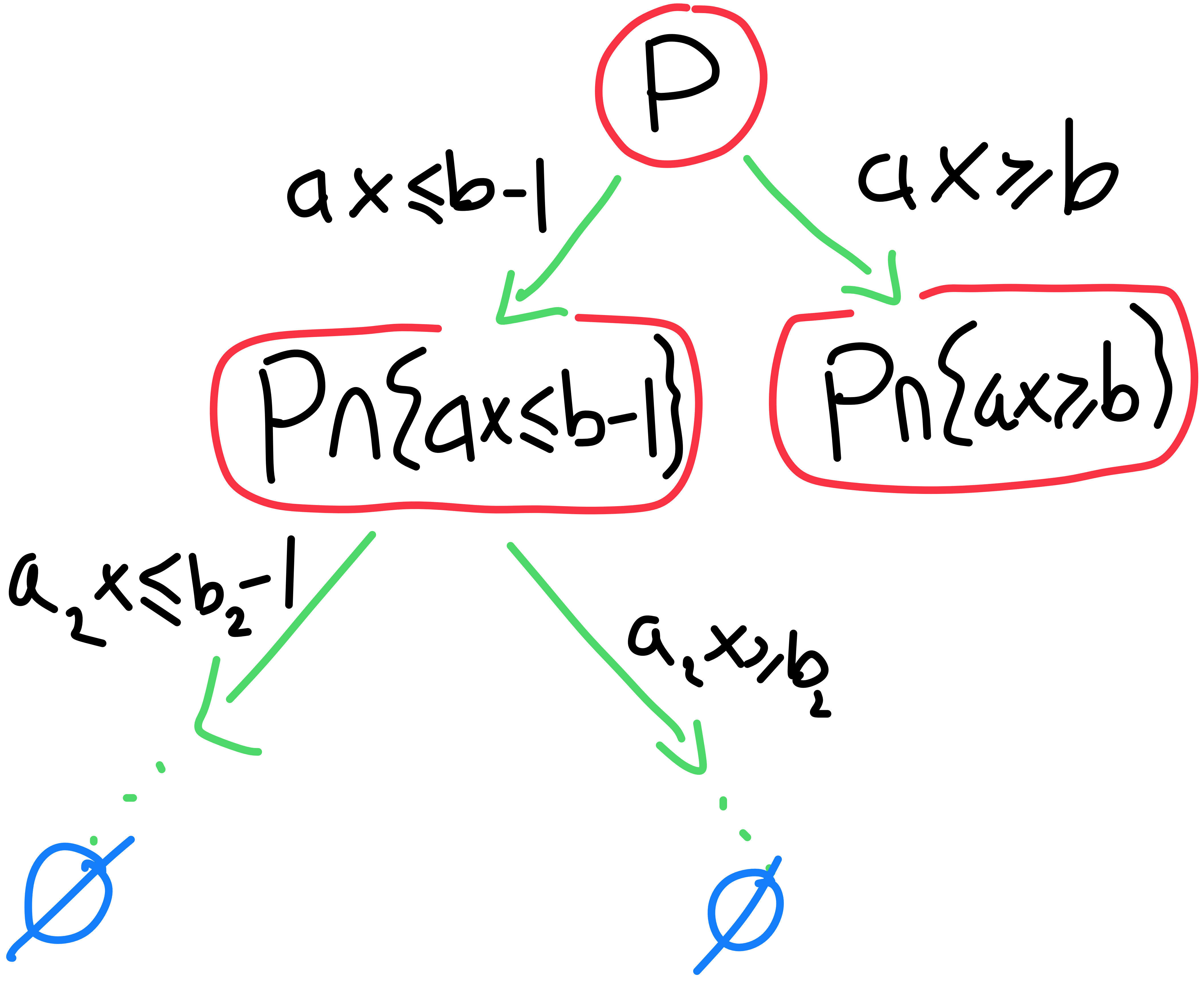

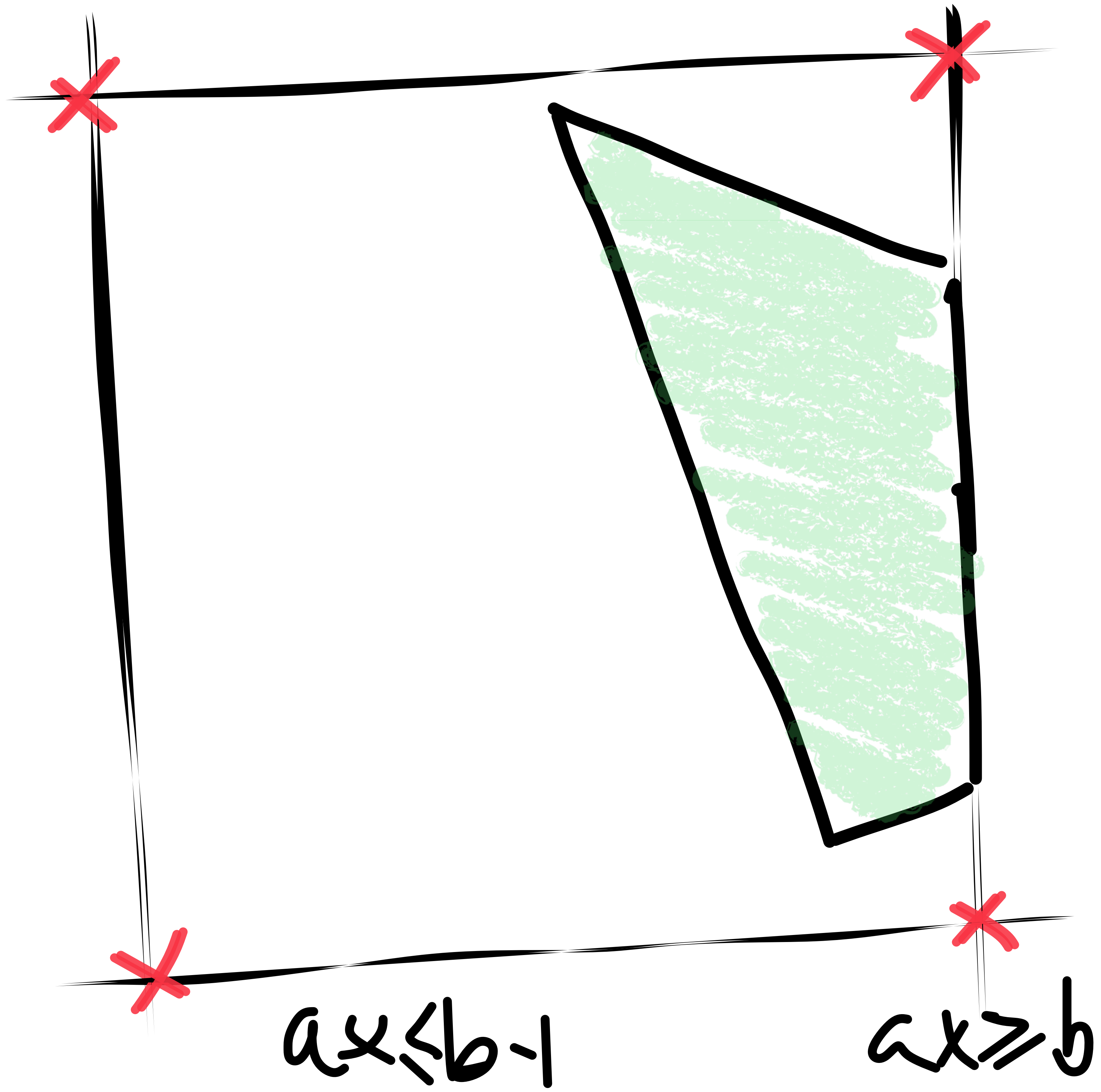

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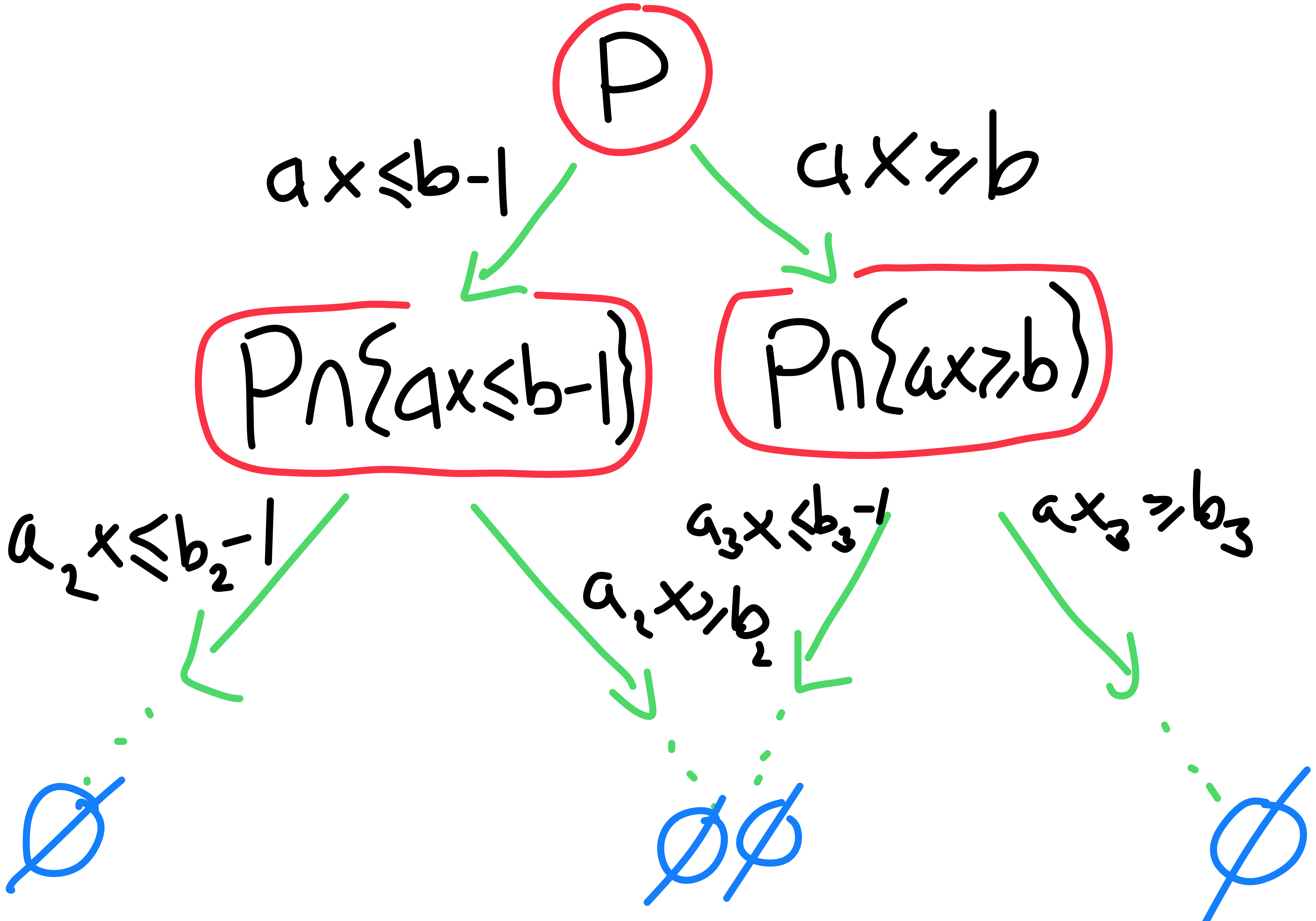

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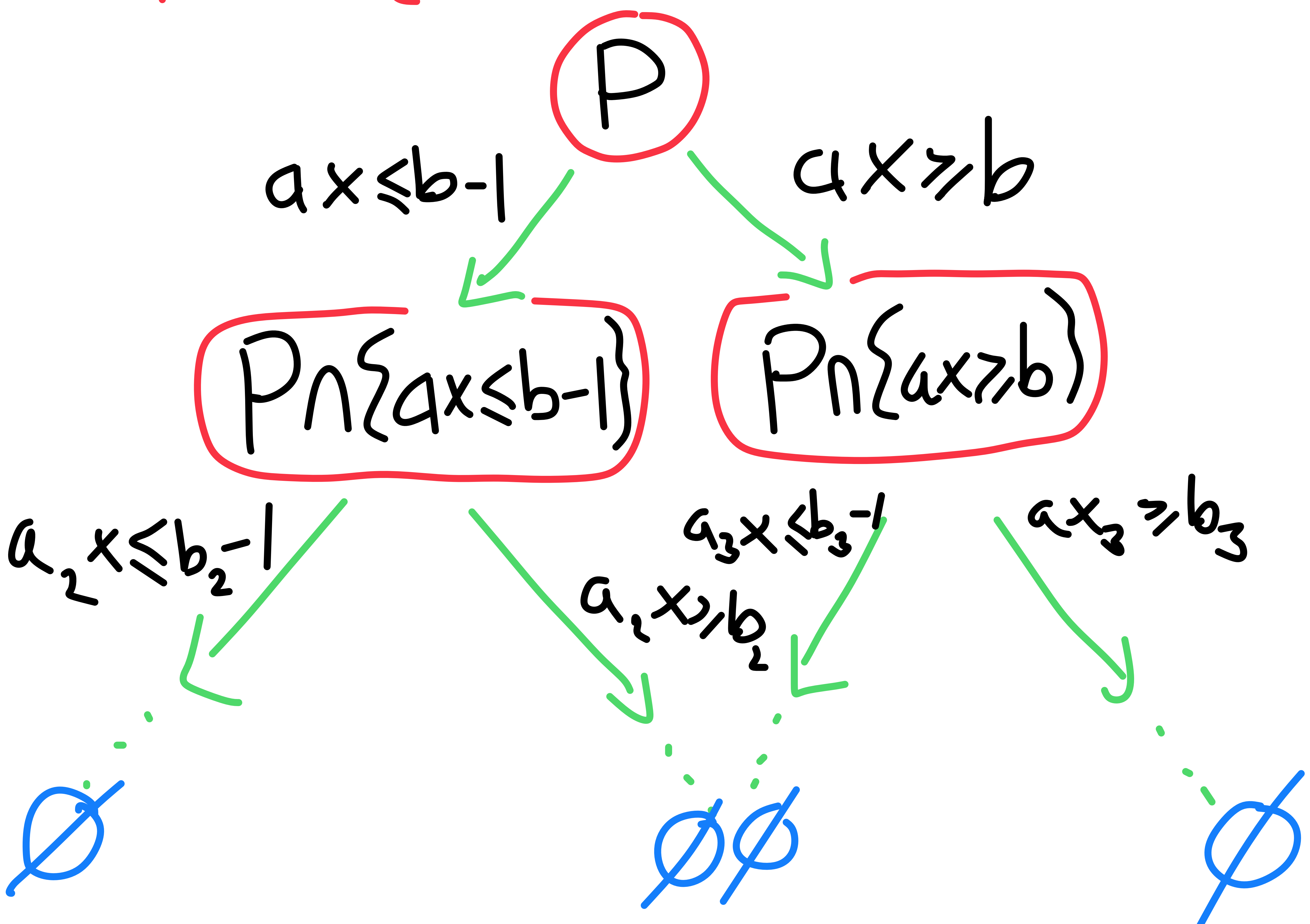

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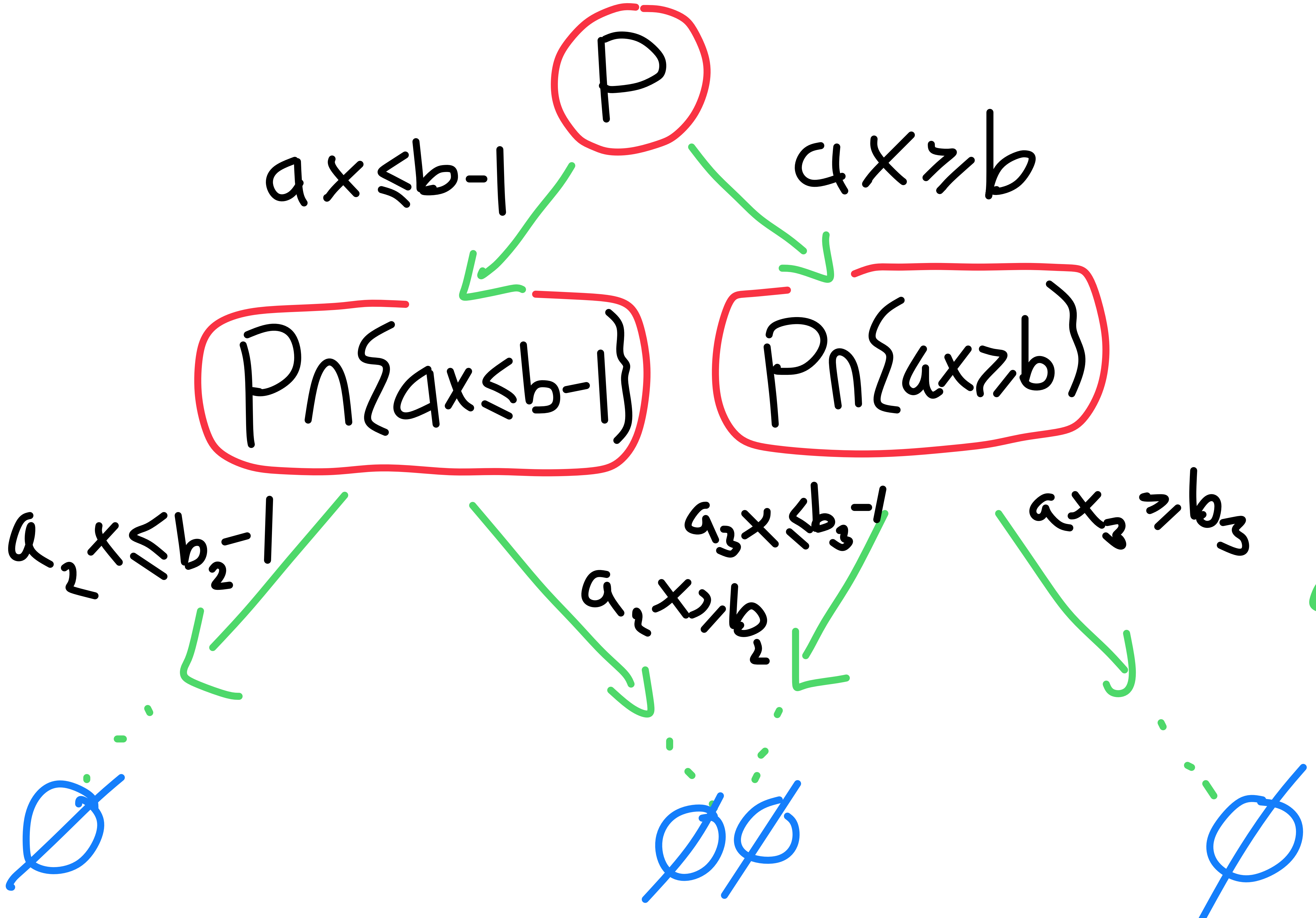

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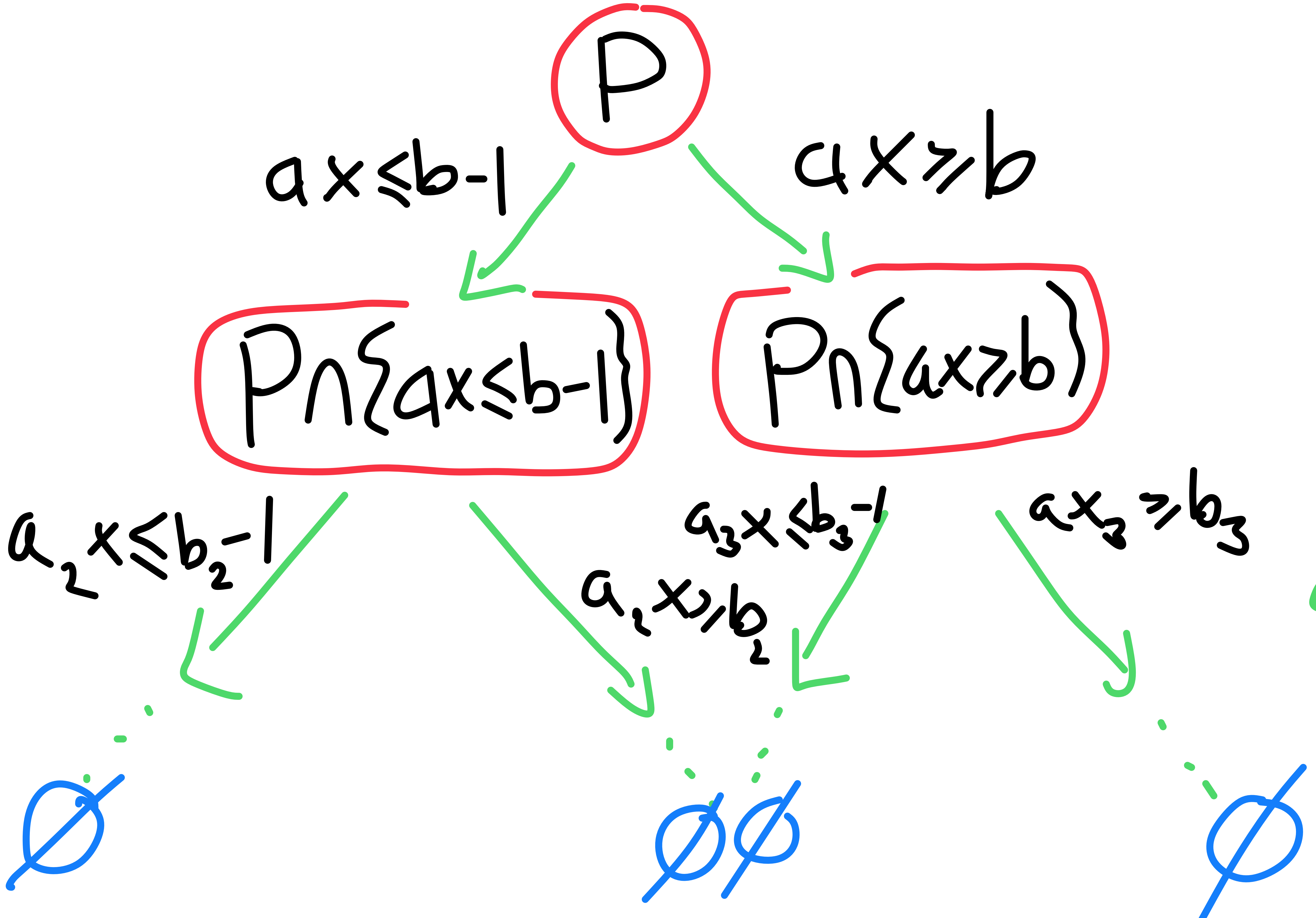

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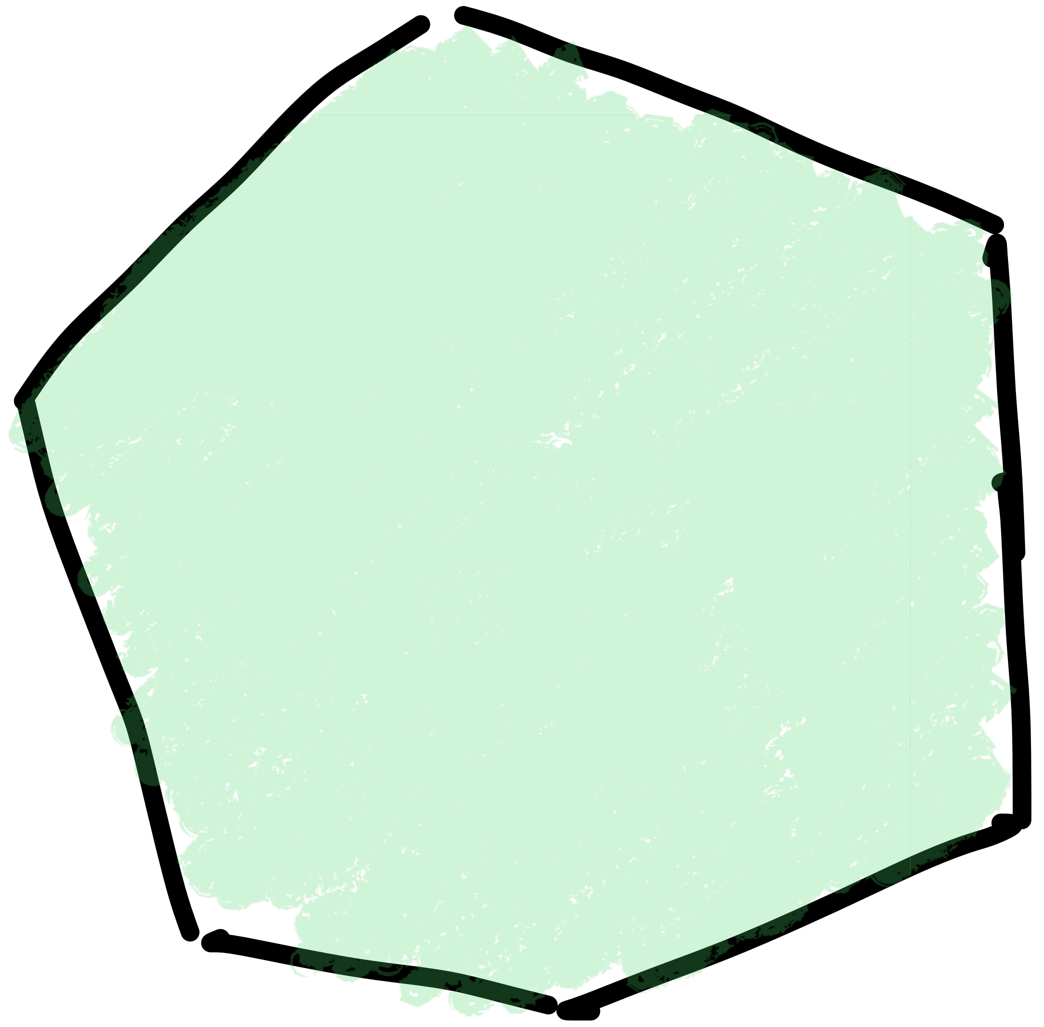

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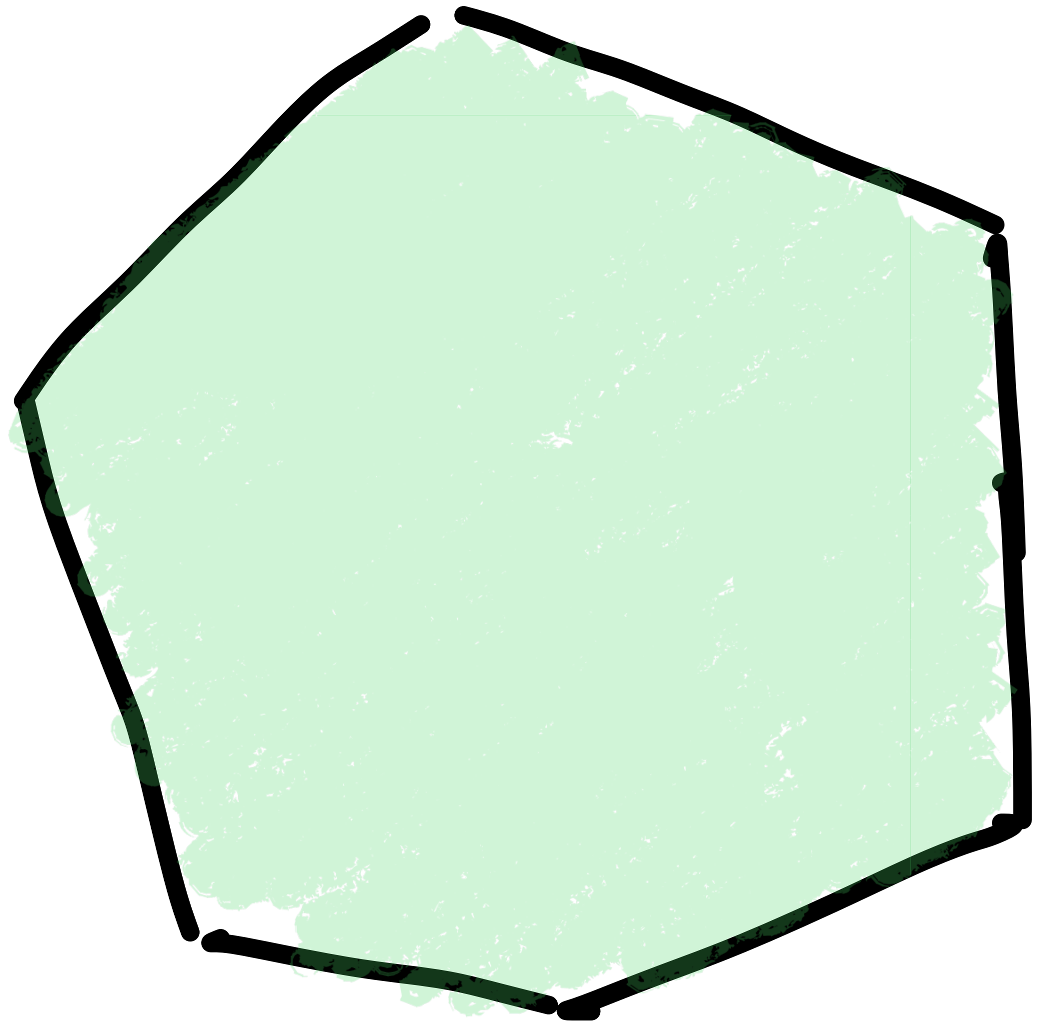

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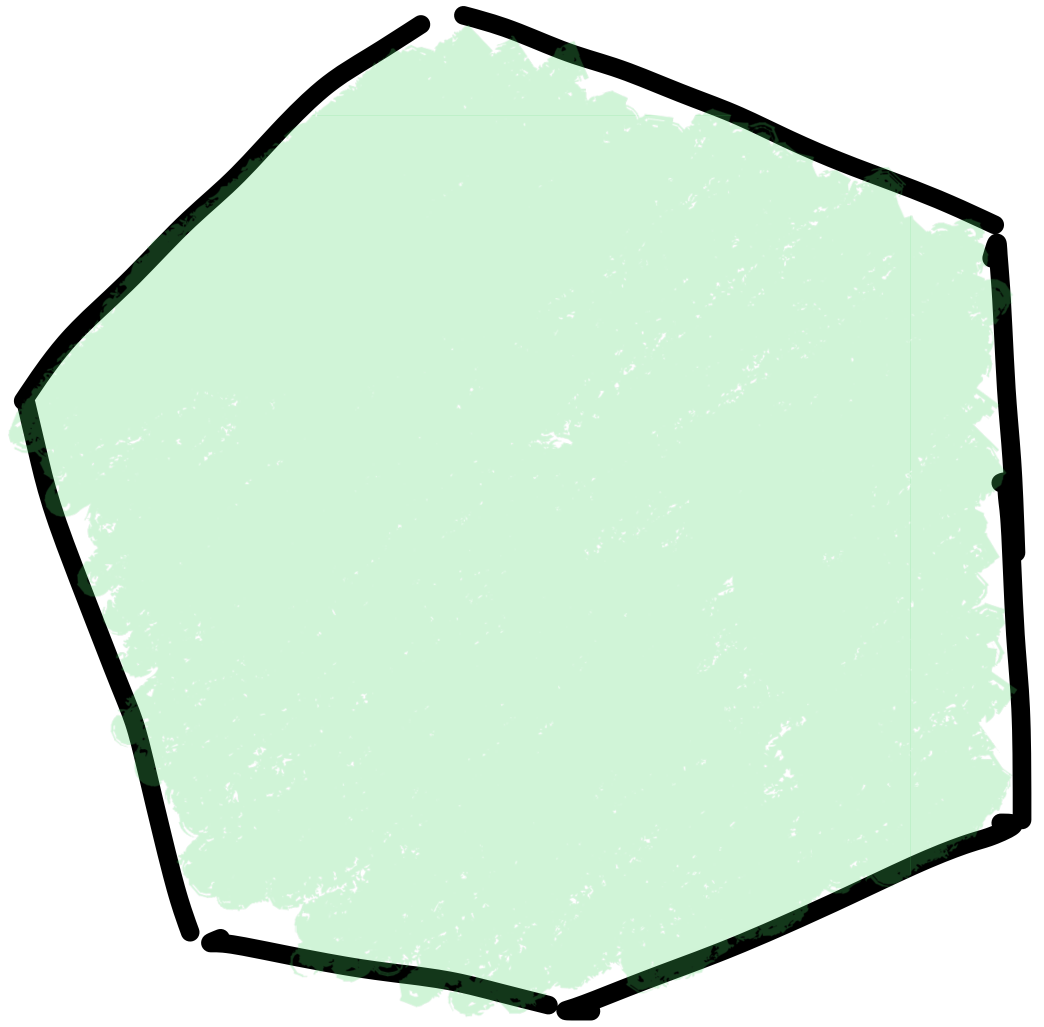

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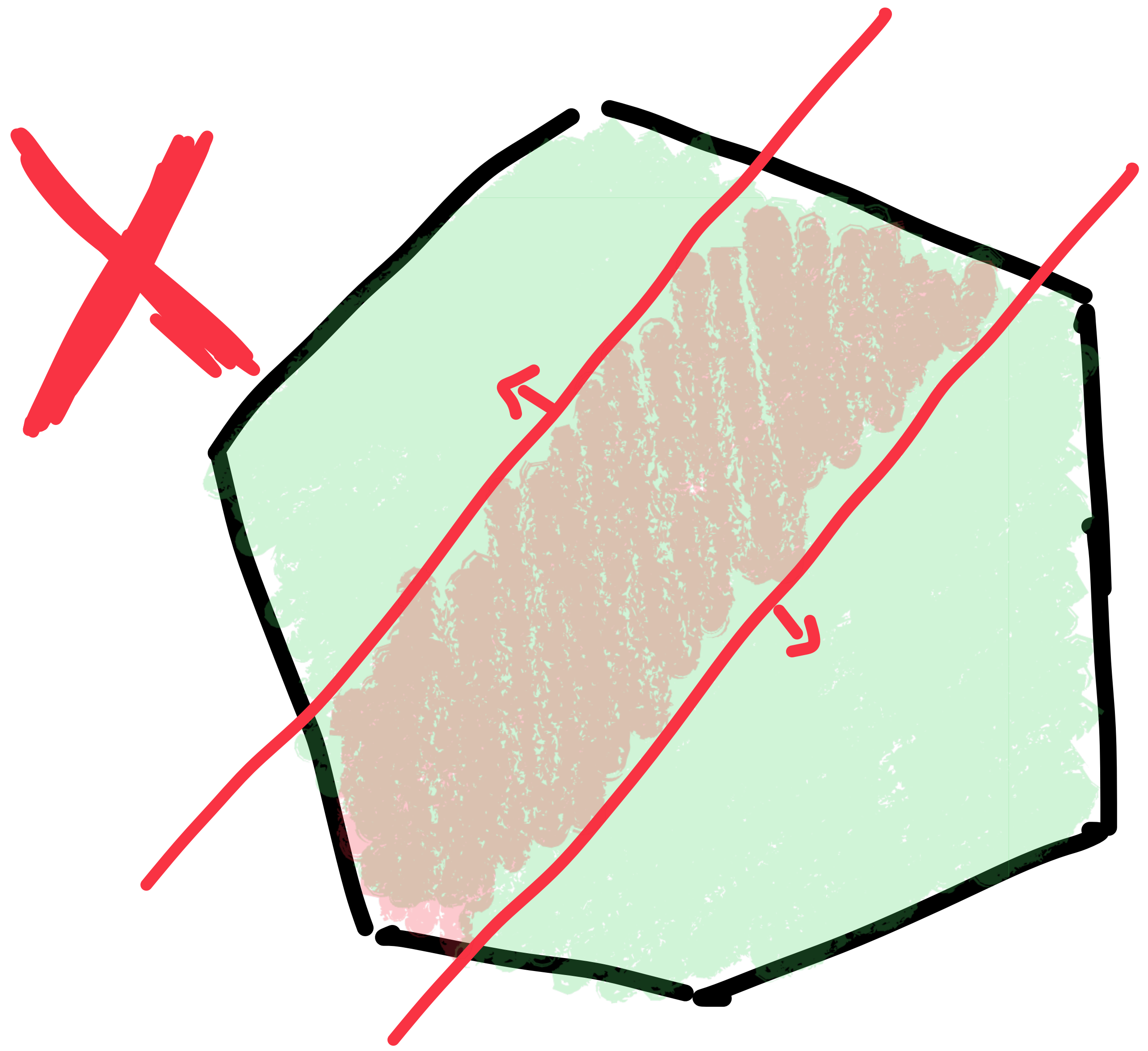

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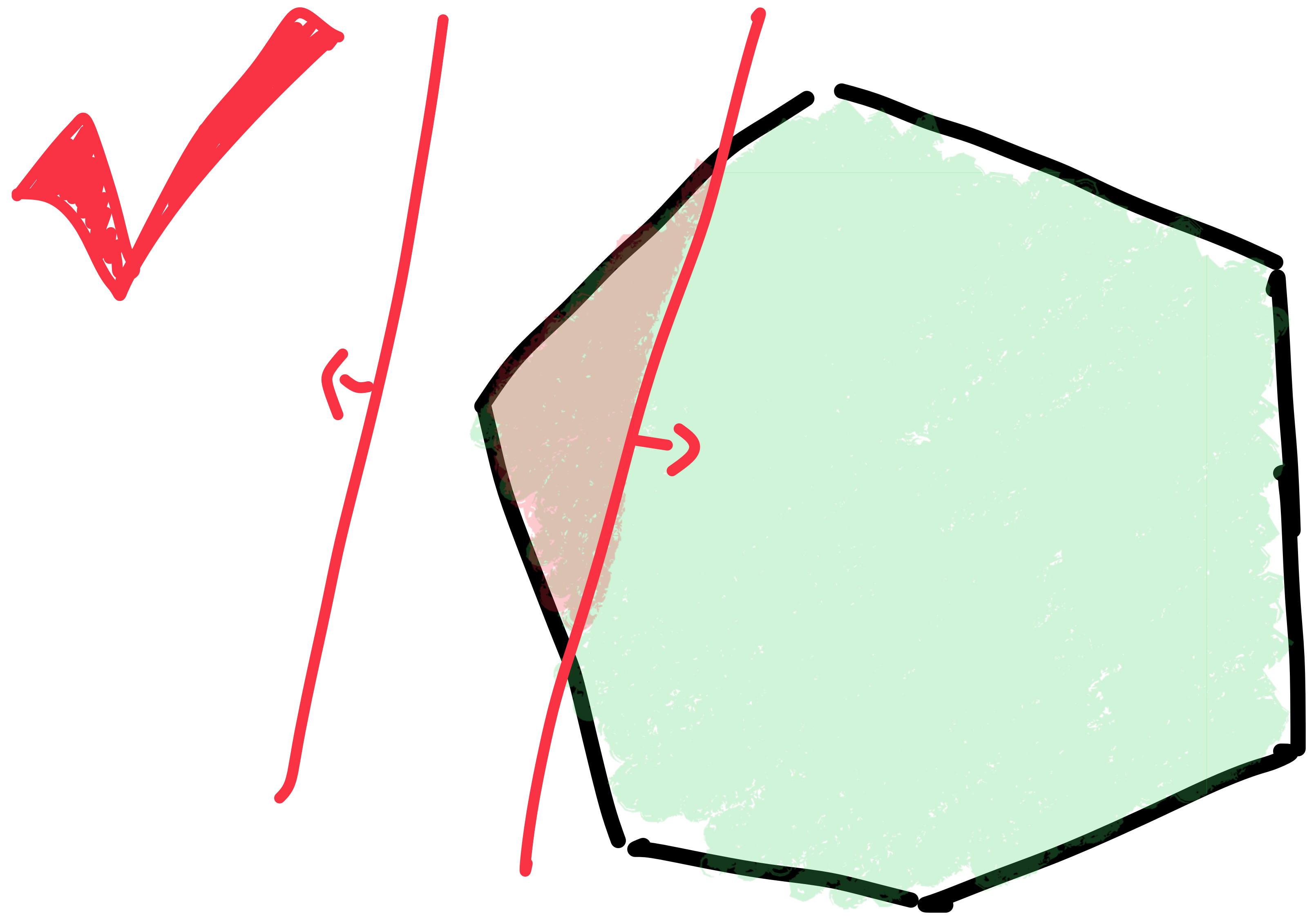

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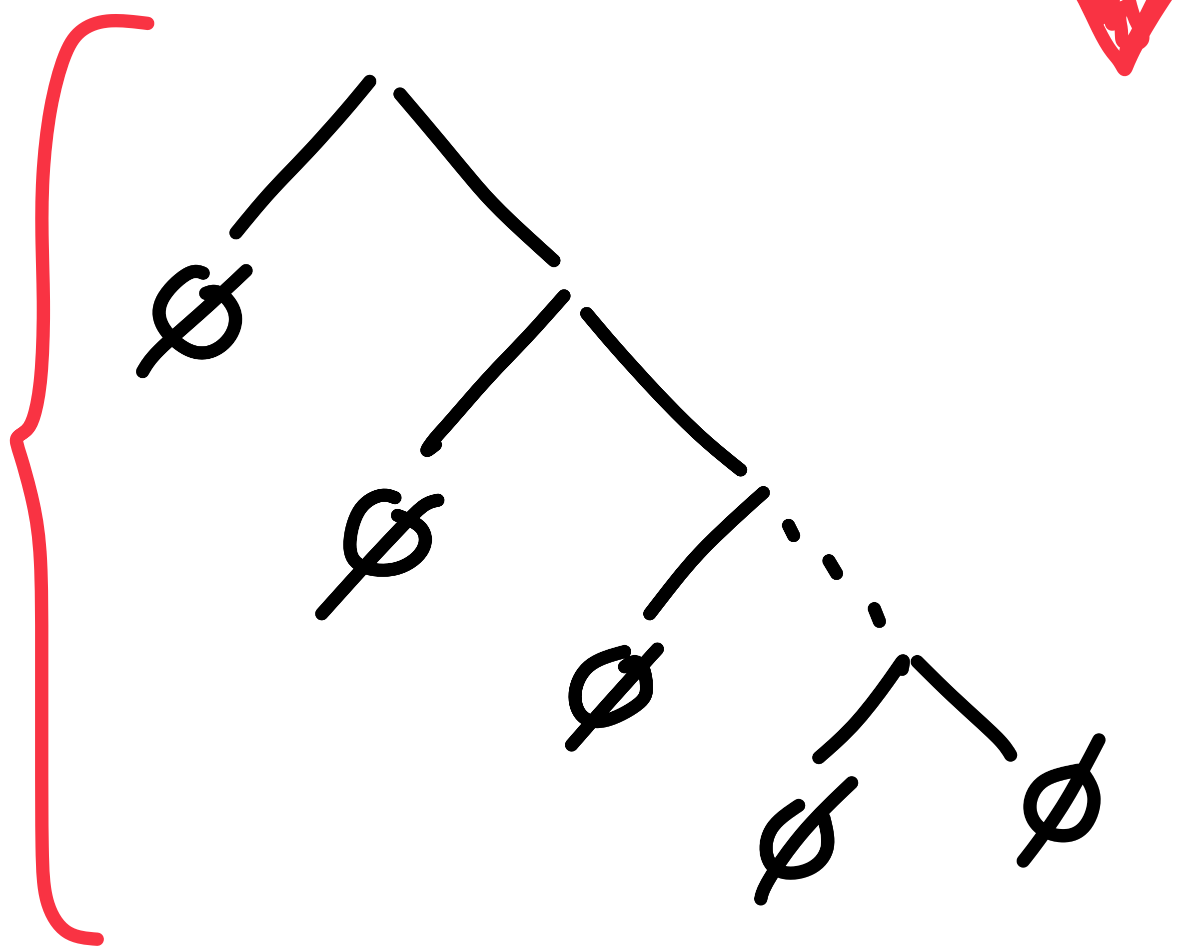
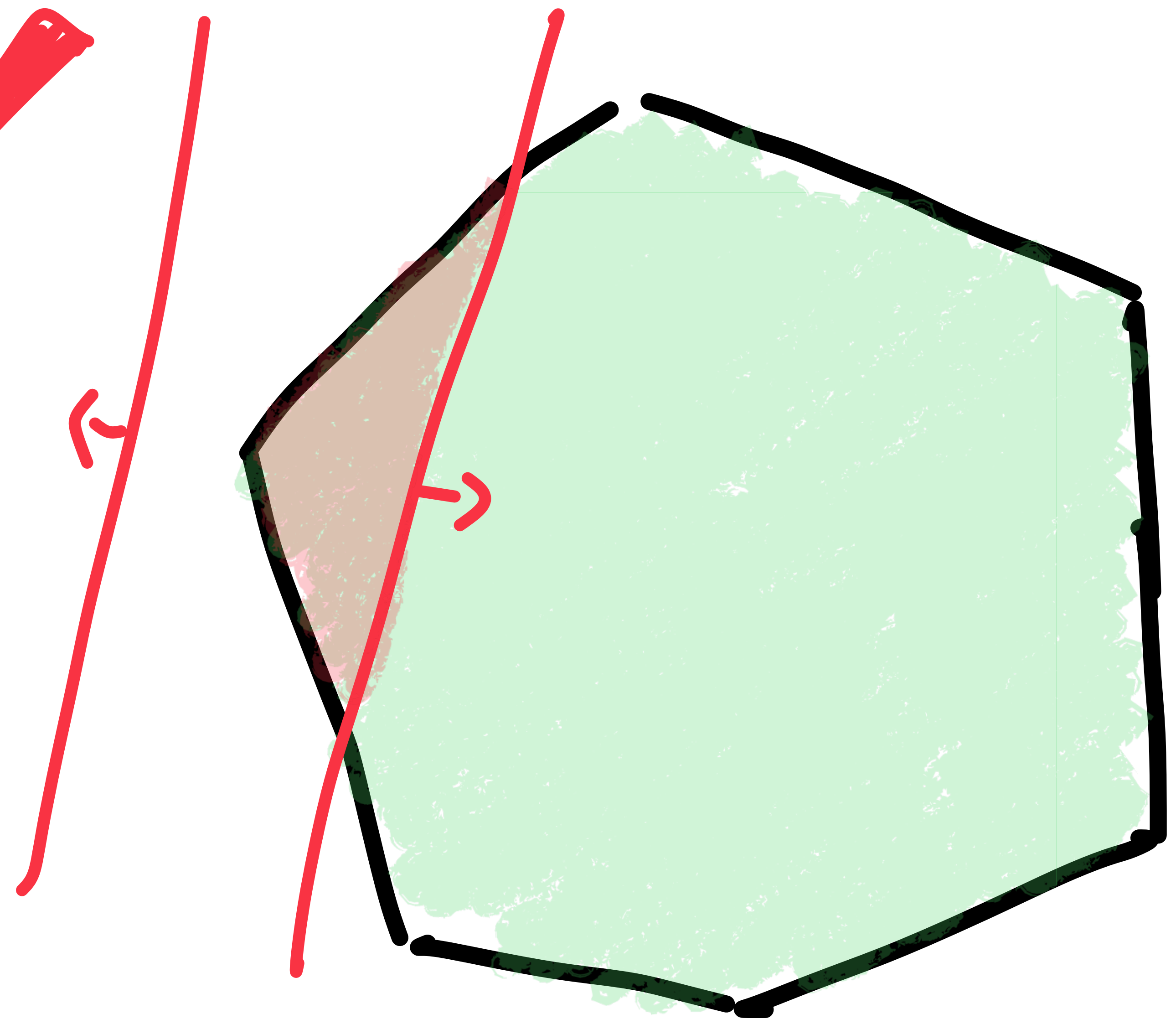
Pathlike SP Proof





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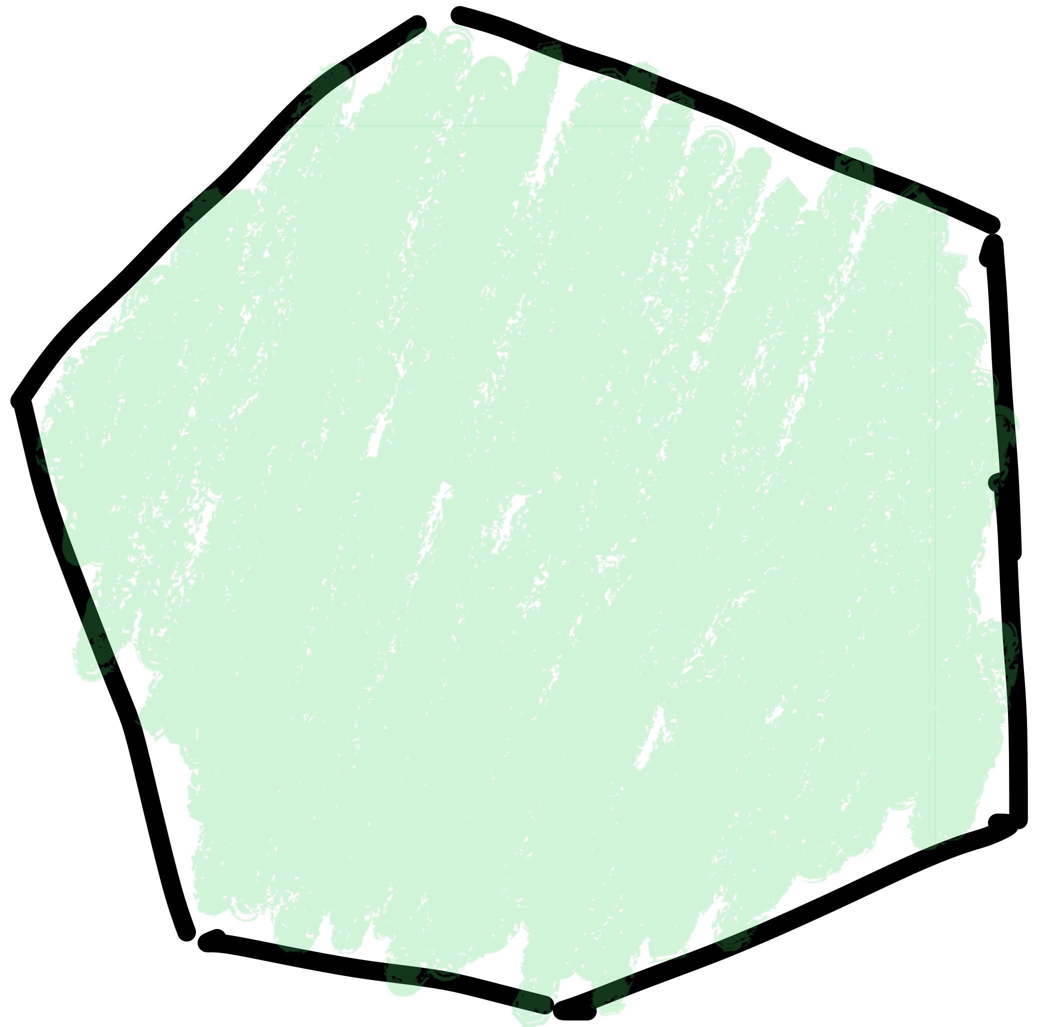

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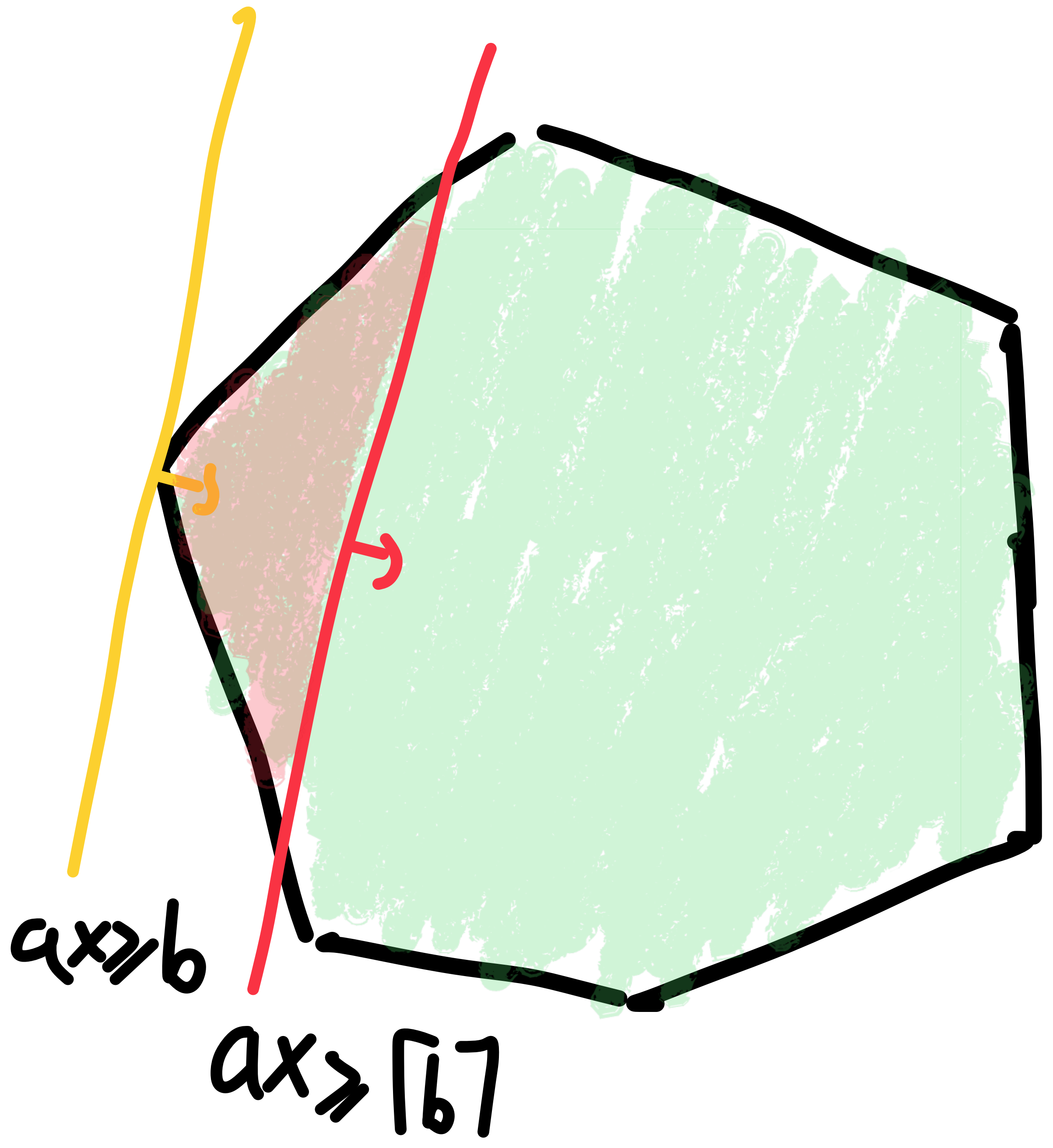
$ax \ge \lceil b \rceil$ is a CG-cut for P if $ax \ge b$ is valid for P



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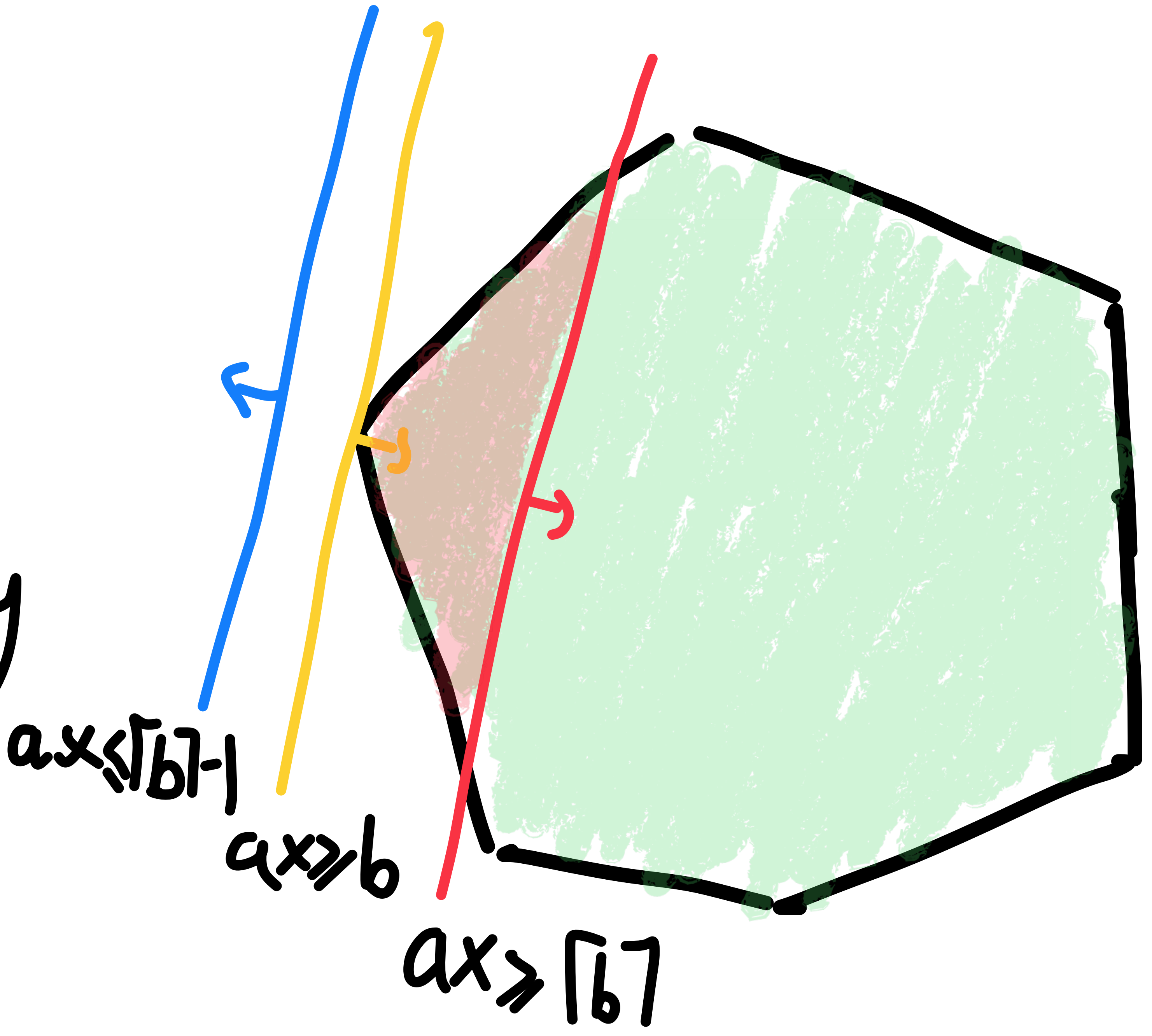

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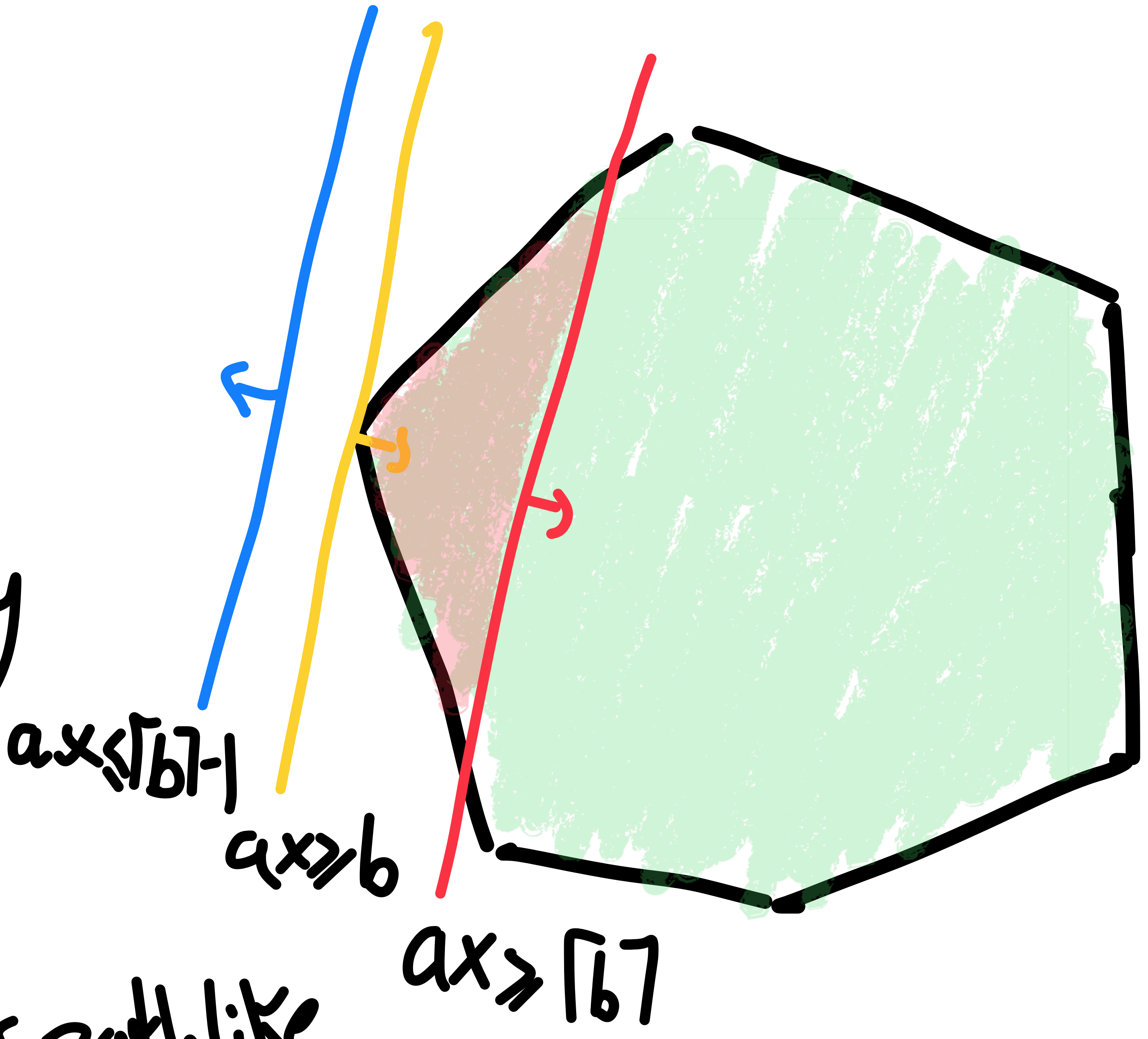

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Coefficients in queries are quasi-poly bounded

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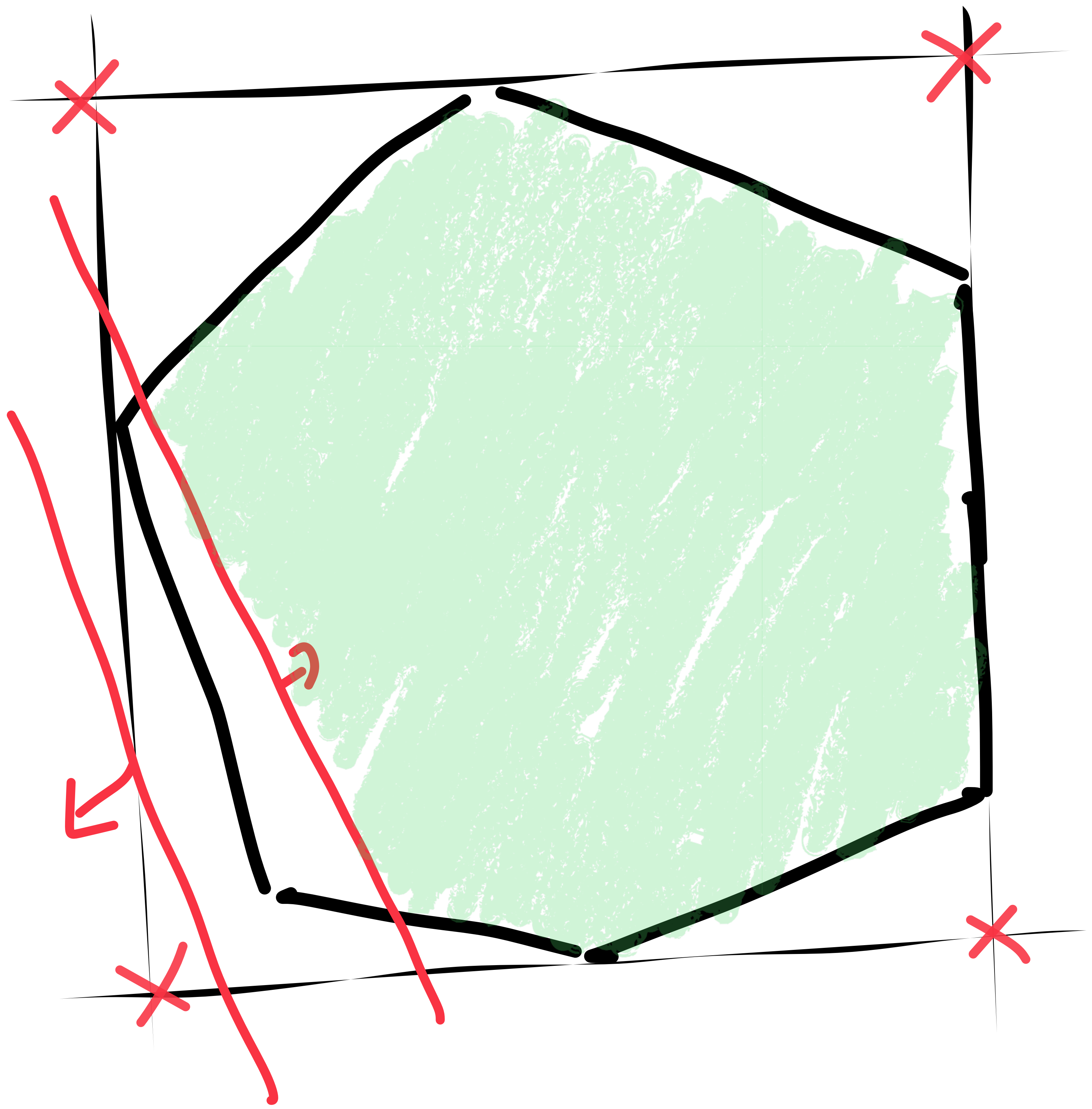

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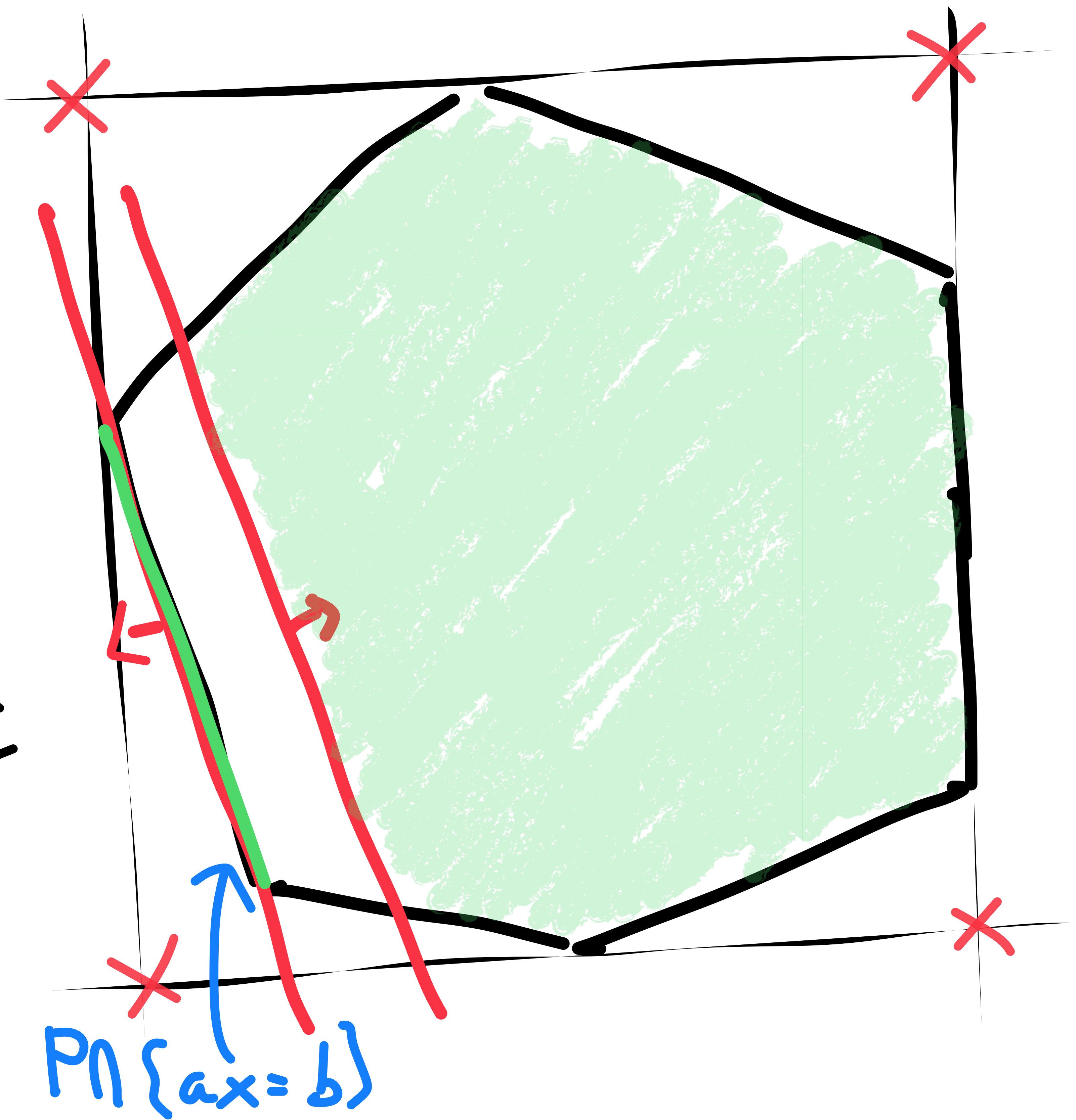

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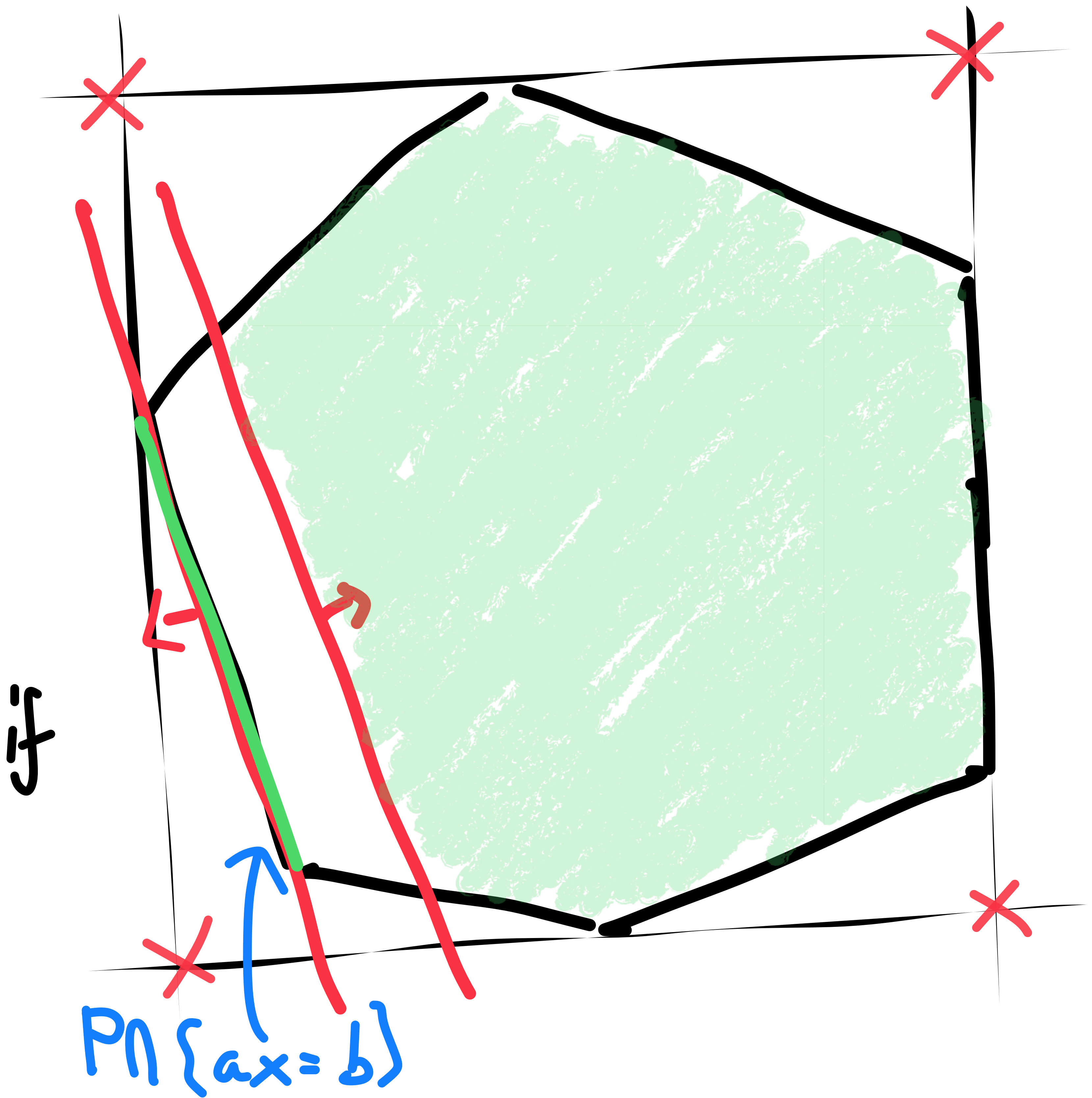

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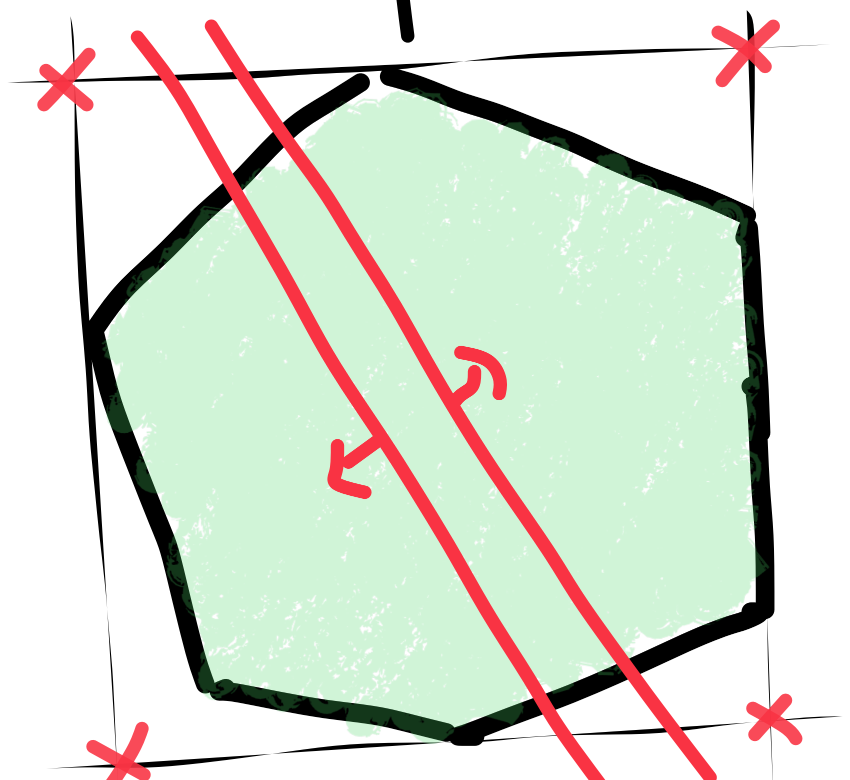

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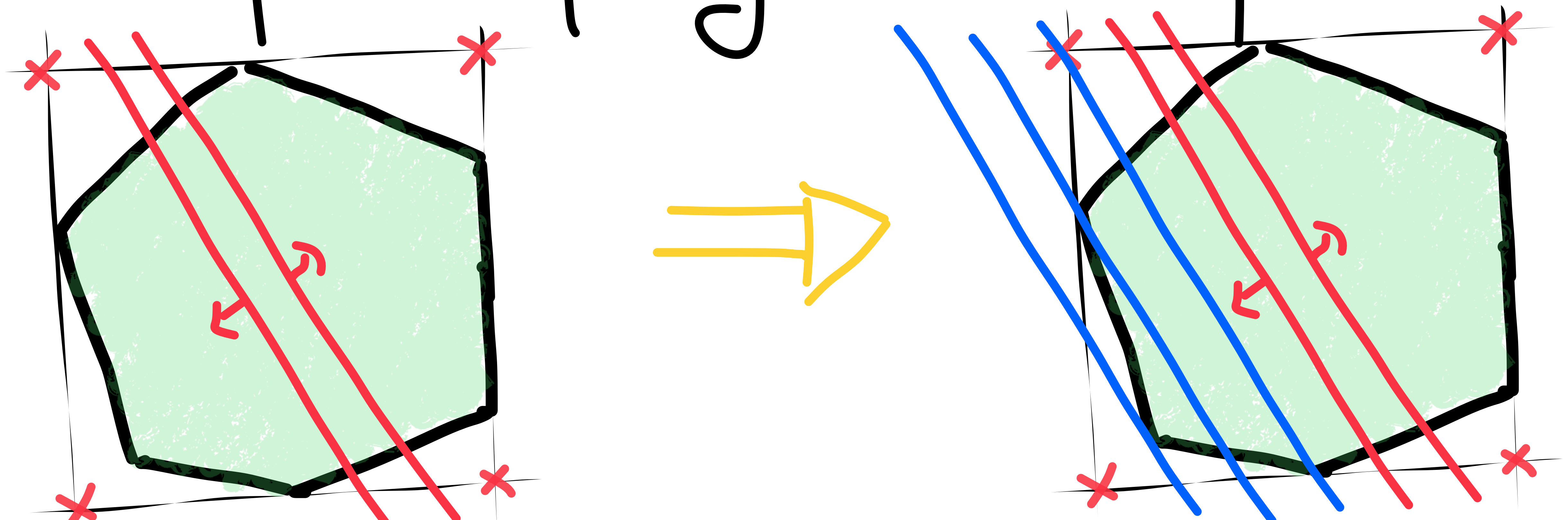

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Cor: Cutting Planes has quasi-poly size proofs of any system of linear equations over a finite field

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- Both (ours and [DT20]) CP proofs of Tseitin are quasi-polynomially deep

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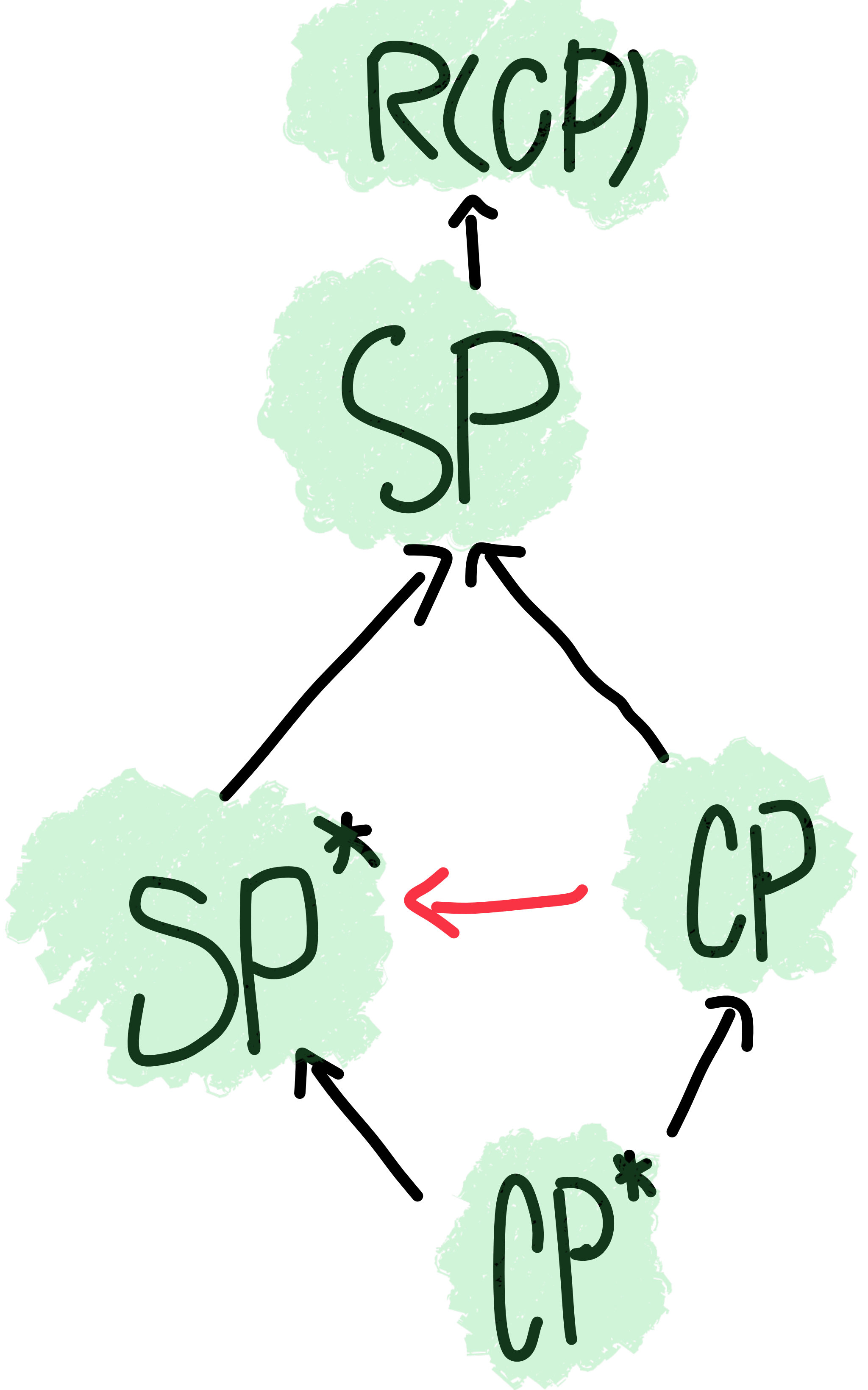
Thm: Tseitin requires $\Omega(n)$ depth to refute in Semantic CP

Open Problems

- Can CP p-simulate SP* ?



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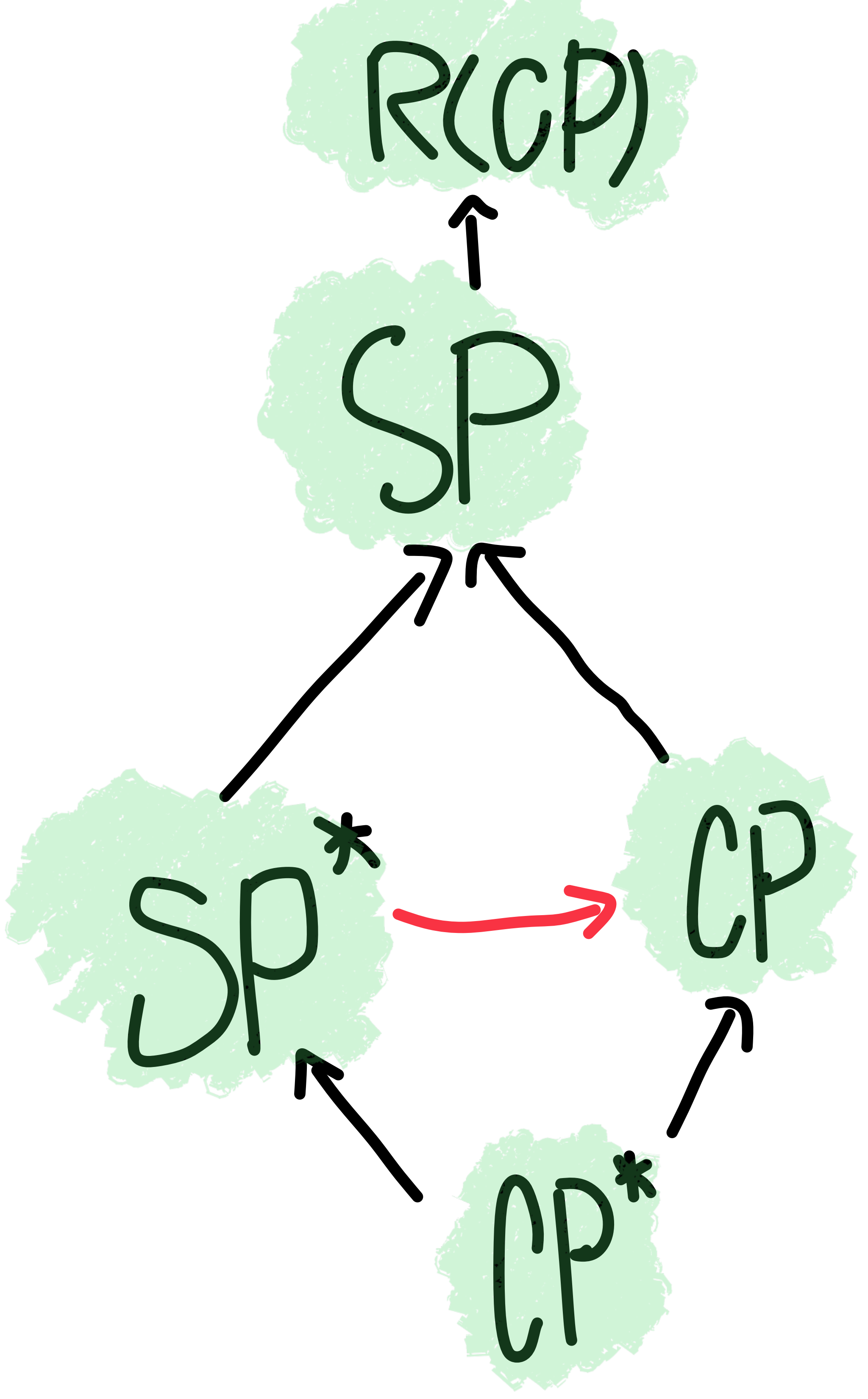
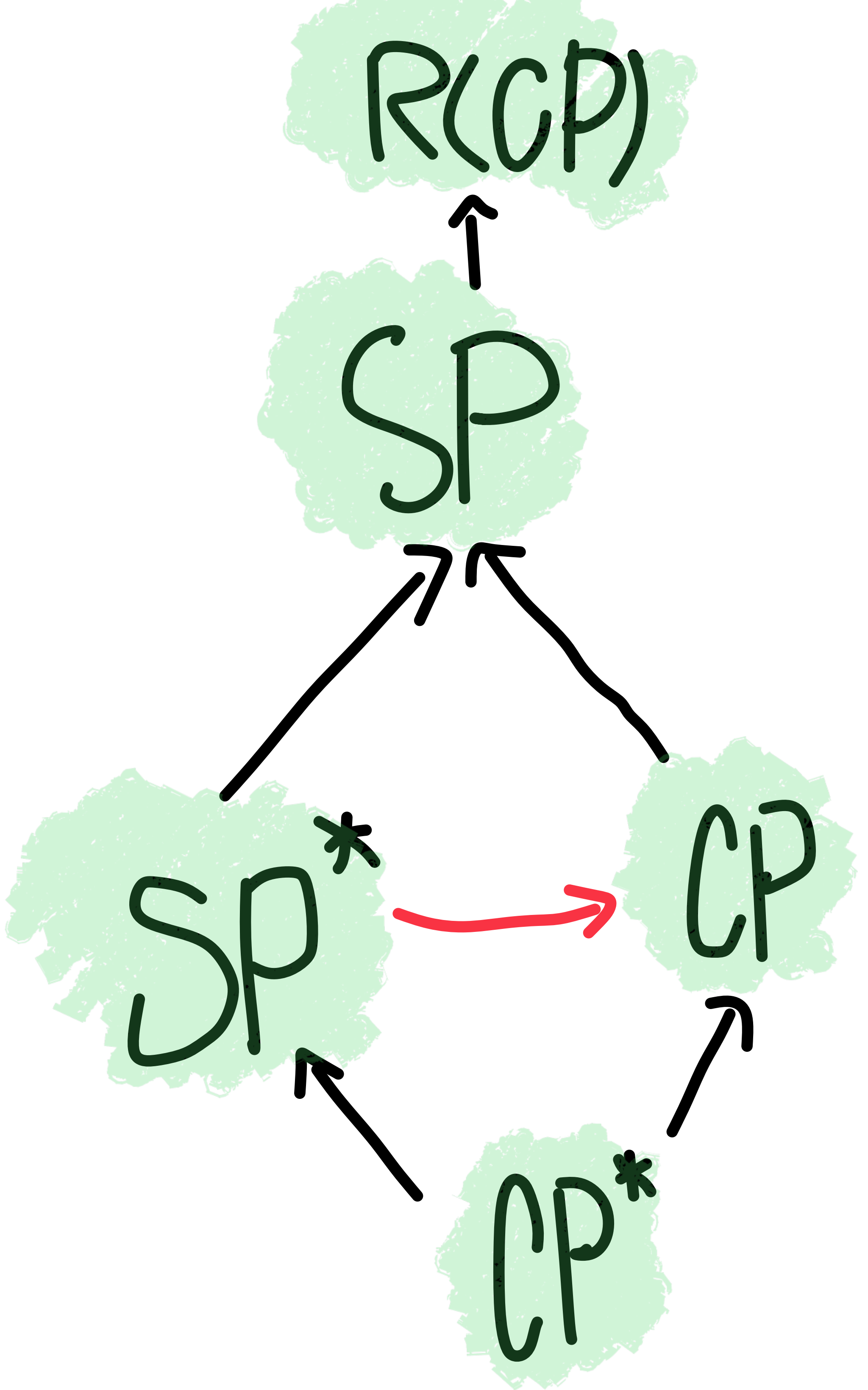
- Can CP (quasipolynomially) simulate SP?



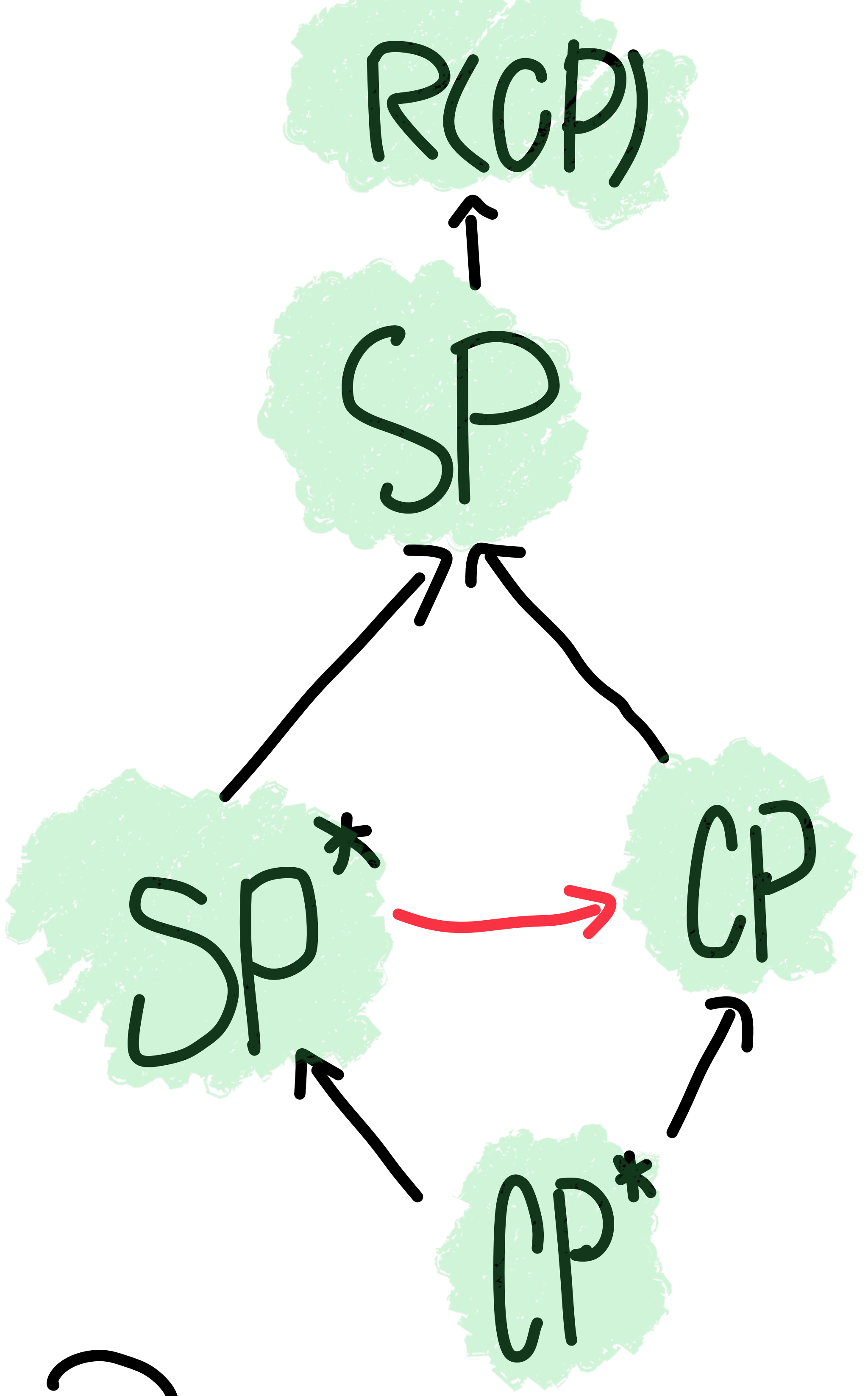
Open Problems

- Can CP p-simulate SP^*?
- Can CP (quasipolynomially) simulate SP?
- Can CP^* (quasipolynomially) simulate SP^*?



Open Problems

- Can CP p-simulate SP*?
- Can CP (quasipolynomially) simulate SP?
- Can CP* (quasipolynomially) simulate SP*?
- Can SP or CP simulate dag-like SP (R(CP))?



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- Can CP^* (quasipolynomially) simulate SP^*?
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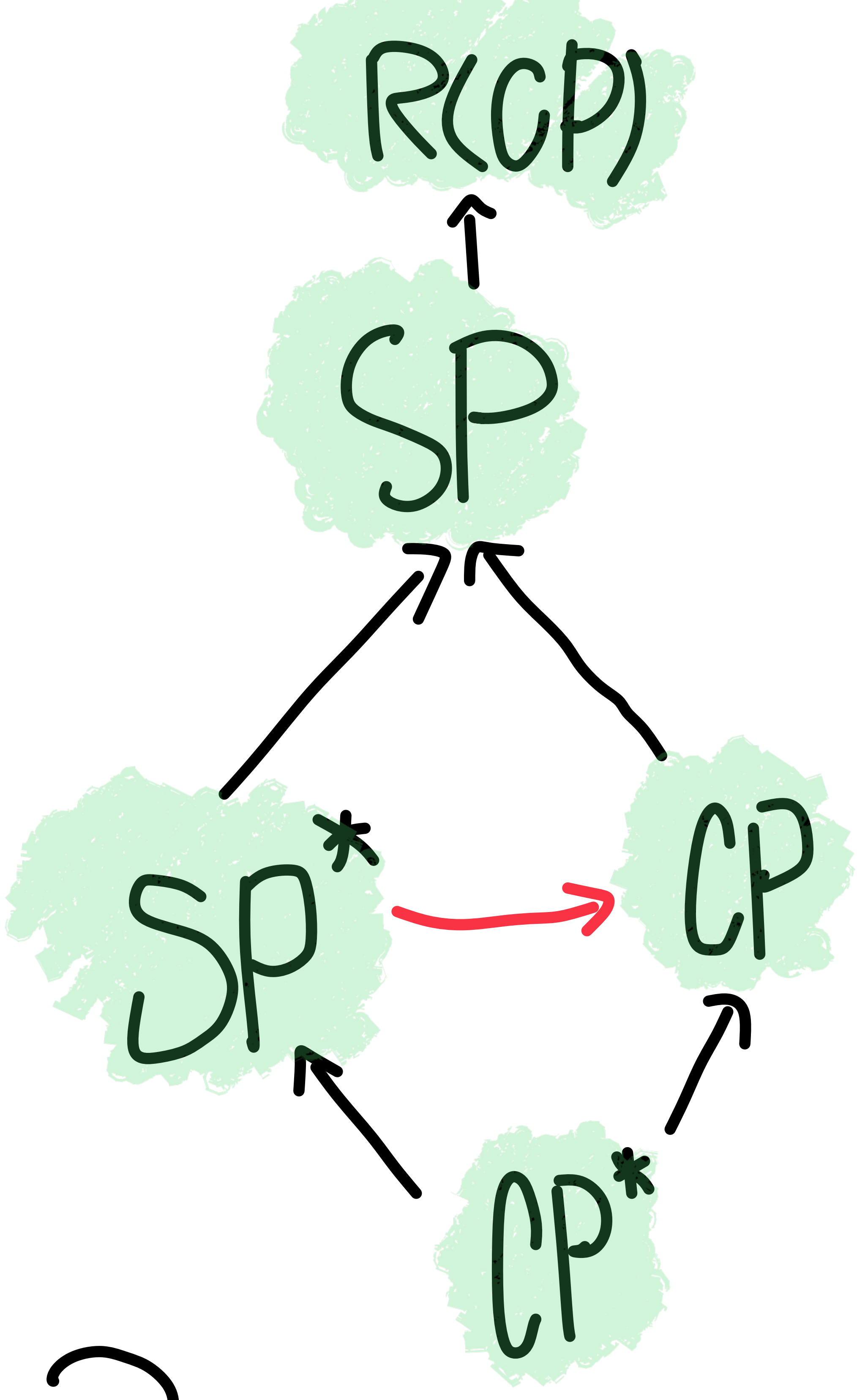
  - → CP cannot p-simulate R(CP) [ABE02]



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- Resolve the Conjecture